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Quotients Stein via la théorie de Patterson-Sullivan et conjecture de Shafarevich

Stein quotients via Patterson-Sullivan theory and the Shafarevich conjecture

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Résumé

Cette thèse est consacrée à l'étude de l'existence de fonctions holomorphes sur certaines variétés complexes et, plus précisément, à la recherche de conditions garantissant que des quotients de variétés kählériennes complètes à courbure négative ou nulle sont des variétés de Stein. Les variétés kählériennes complètes simplement connexes à courbure négative ou nulle, comme l'espace hyperbolique complexe, sont des variétés de Stein. Une question centrale est de savoir, lorsqu'un groupe discret et sans torsion agit proprement discontinûment par biholomorphismes sur un tel espace, si le quotient conserve la propriété d'être de Stein.

Le premier pan de nos résultat est consacré aux quotients de la boule complexe. Nous y étudions les sous-groupes discrets et sans torsion de $\mathrm{PU}(n,1)$. Pour les sous-groupes élémentaires, nous donnons une caractérisation complète des cas où le quotient est de Stein, généralisant des résultats antérieurs de C. Miebach. Pour les sous-groupes géométriquement finis, nous montrons que le quotient est holomorphiquement convexe si et seulement si les quotients associés à chaque sous-groupe parabolique maximal le sont également. Plusieurs résultat précisent le lien entre convexité holomorphe, existence de sous-variétés analytiques compactes et invariants dynamiques, notamment l'exposant critique du groupe. Certaines conjectures de Dey et Kapovich sont ainsi partiellement résolues à l'aide de la théorie de Patterson–Sullivan.

Un deuxième pan, réalisé en collaboration avec C. Davalo, concerne les quotients d'espaces hermitiens symétriques de rang supérieur. Nous introduisons un critère fondé sur l'entropie du groupe relativement à la racine associée au bord de Shilov de l'espace symétrique en question. Notre résultat principal montre que si cette entropie est inférieure à 1, le quotient ne contient aucune sous-ensemble analytique compact de dimension positive. De plus, lorsque l'entropie est égale à 1, de tels sous-ensembles existent si et seulement si le groupe considéré est le groupe fondamental d'une surface et préserve un disque complexe totalement géodésique. Ce travail repose sur une analyse des fonctions de Busemann et sur la théorie de Patterson-Sullivan en rang supérieur.

Enfin, dans une troisième partie, nous nous intéressons aux compactifications toroïdales de quotients de la boule, qui sont des variétés projectives obtenues en compactifiant certains quotients non compacts de volume fini de la boule. Nous démontrons qu'il n'existe aucune métrique kählérienne à courbure sectionnelle négative ou nulle sur ces compactifications, mais qu'il est possible de construire sur certaines de ces variétés des métriques à courbure bisectionnelle holomorphe quasi-négative ce qui répond à une question posée par Diverio. Enfin, dans le cas des réseaux arithmétiques, nous prouvons que le revêtement universel de certaines de ces compactifications est une variété de Stein, ce qui généralise en toute dimension un résultat qui n'était connu jusque-là qu'en dimension deux. Ceci établit, pour ces exemples, la conjecture de Shafarevich, qui prédit que le revêtement universel d'une variété kählérienne compacte est toujours holomorphiquement convexe.

Abstract

This thesis is devoted to the study of holomorphic functions on certain complex manifolds, with a particular focus on identifying conditions under which quotients of complete Kähler manifolds of nonpositive curvature are Stein. Complete and simply connected Kähler manifolds of nonpositive curvature, such as complex hyperbolic space, are examples of Stein manifolds. One central question addressed here is whether the quotient of such a space by a discrete and torsion-free group acting properly discontinuously by biholomorphisms is still a Stein manifold.

The first part of our results concerns quotients of the complex ball, through the study of discrete and torsion-free subgroups of $\mathrm{PU}(n,1)$. For elementary subgroups, we obtain a complete characterization of when the quotient is Stein, extending earlier work of Miebach. For geometrically finite subgroups, we prove that the quotient is holomorphically convex if and only if the quotients associated with each maximal parabolic subgroup are also holomorphically convex. Several results further clarify the interplay between holomorphic convexity, the existence of compact analytic subvarieties, and dynamical invariants such as the critical exponent of the group. In this framework, several conjectures of Dey and Kapovich are partially resolved, using techniques from Patterson–Sullivan theory.

A second line of work, carried out in collaboration with C. Davalo, investigates quotients of higher rank Hermitian symmetric spaces. We introduce a criterion based on the entropy of the group relatively to the root associated with the Shilov boundary of the symmetric space. Our main result establishes that when this entropy is strictly less than one, the quotient admits no compact analytic subvarieties of positive dimension. Moreover, when the entropy equals one, there are such subvarieties in the quotient if and only if the group is the fundamental group of a surface and preserves a totally geodesic complex disk. This analysis relies on a refined study of Busemann functions and Patterson–Sullivan theory in the higher rank framework.

Finally, in a third part, we study toroidal compactifications of ball quotients, which are projective varieties obtained by compactifying certain nonuniform quotients of the complex ball. We show that these compactifications do not admit Kähler metrics of nonpositive sectional curvature, but that some of them admit a Kähler metric of quasi-negative holomorphic bisectional curvature, thereby answering a question raised by Diverio. Finally, in the arithmetic setting, we prove that the universal cover of sufficiently deep toroidal compactifications is Stein, extending to all dimensions a result previously known only in complex dimension two. This establishes, for these examples, the Shafarevich conjecture, which predicts that the universal cover of any compact Kähler manifold is holomorphically convex.

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L'image est une enluminure médiévale illustrant la citation attribuée à Bernard de Chartres, « Nous sommes comme des nains assis sur des épaules de géants ».

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[1]Louis Guillaume, *La Vitre*.

Contents

Introduction en français

Cette thèse a pour thématique l'existence de fonctions holomorphes sur certaines variétés complexes. Un ouvert $X \subset \mathbb{C}^n$ admet, par restrictions de fonctions définies sur $\mathbb{C}^n$, suffisamment de fonctions holomorphes $f\colon X \to \mathbb{C}$ pour que la propriété de séparabilité suivante soit vérifiée :

$$\forall x \neq y \in X, \exists f\colon X \to \mathbb{C} \text{ holomorphe telle que } f(x) \neq f(y). \tag{S}$$

Tous les ouverts de $\mathbb{C}^n$ ne vérifient cependant pas la propriété suivante

$$\forall (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ sans accumulation}, \exists f\colon X \to \mathbb{C} \text{ holomorphe telle que } \sup_n |f(x_n)| = +\infty. \tag{HC}$$

Par exemple une boule épointée dans $\mathbb{C}^2$ ne vérifie pas cette propriété, puisque toute fonction holomorphe définie sur une boule épointée de dimension au moins 2 se prolonge holomorphiquement en 0 (voir par exemple [Kra01, Theorem 1.2.6]).

Plus généralement, une variété complexe X est appelée *holomorphiquement convexe* si elle vérifie la propriété (**HC**). Si de plus les fonctions holomorphes sur X *séparent les points*, au sens de la propriété (**S**), on dit que X est une *variété de Stein*. Les variétés de Stein admettent de nombreuses définitions équivalentes, comme nous le verrons dans la Section 1.1.4. Tout au long de ce travail, nous utiliserons la caractérisation suivante, qui est une conséquence de l'existence de la réduction de Remmert : une variété X est de Stein si et seulement si elle est holomorphiquement convexe et ne contient pas de sous-ensemble analytique compact de dimension positive (voir Définition 1.15 et les références dans la Section 1.1.4).

Une variété compacte de dimension positive ne vérifie jamais (**S**) et vérifie trivialement (**HC**), mais pour les variétés non compactes les propriétés (**S**) et (**HC**) sont indépendantes et peuvent être, ou non, réalisées. Un revêtement, éventuellement ramifié, d'une variété de Stein est de Stein [Le 76, Théorème 1], mais le quotient d'une variété de Stein n'est pas en général de Stein. Un des fils directeurs de cette thèse est le suivant :

> Étant donnée une variété complexe simplement connexe X, obtenir des critères portant sur les groupes Γ agissant librement et proprement discontinûment par biholomorphismes sur X qui permettent d'affirmer que X/Γ est une variété de Stein.

Pour qu'au moins un quotient X/Γ soit Stein, il faut que X soit une variété de Stein d'après le résultat rappelé ci-dessus. Un résultat de Greene et Wu ([GW79, Proposition 1.17], voir aussi l'exemple de la Section 1.1.4) affirme qu'une variété kählérienne complète simplement connexe à courbure sectionnelle négative ou nulle est une variété de Stein. Nous restreindrons notre champ d'étude aux quotients de ces variétés, et ajouterons l'hypothèse que les groupes agissent par *isométries* biholomorphes sur ces variétés. Dans ce contexte, nous restreignons encore notre étude à deux types de variétés kählériennes complètes simplement connexe, d'une part celles à courbure sectionnelle inférieure ou égale à -1, et d'autre part les espaces symétriques hermitiens de type non compact.

L'intersection de ces deux types de variété donne exactement la famille des espaces hyperboliques complexes $\mathbb{H}^n_{\mathbb{C}}$, qui sont des variétés biholomorphes à la boule unité de $\mathbb{C}^n$, munies d'une métrique kählérienne à courbure sectionnelle variable comprise entre -4 et -1. C'est pour ces espaces que nous obtenons le plus de résultats. Notons dès maintenant que tout biholomorphisme de la boule unité est une isométrie de l'espace hyperbolique complexe, et que le groupe des biholomorphismes de $\mathbb{H}^n_{\mathbb{C}}$ est le groupe $\mathrm{PU}(n,1)$, défini en Section 1.2.1. De plus, un sous-groupe Γ de $\mathrm{PU}(n,1)$ agit librement et proprement discontinûment sur $\mathbb{H}^n_{\mathbb{C}}$ si et seulement s'il est discret et sans torsion. Dans le cas de la boule unité de $\mathbb{C}^n$, ce travail est motivé par des questions posées par Dey et Kapovich dans [DK20], que nous détaillerons plus loin.

Dans cette introduction, nous commençons par décrire les résultats obtenus concernant les quotients de la boule par des sous-groupes discrets et sans torsion de $\mathrm{PU}(n,1)$, qui ont fait l'objet de la prépublication [Sar24]. Certaines de nos techniques de preuves utilisent la théorie de Patterson–Sullivan, et s'adaptent au rang supérieur. Dans la deuxième partie de l'introduction, nous décrivons des résultats non publiés portant sur les quotients d'espaces symétriques hermitiens de rang supérieur, obtenus en collaboration avec Colin Davalo. Enfin, une troisième section concerne les résultats pré-publiés dans [Sar23] qui portent sur le revêtement universel de certaines compactifications toroïdales de quotients de la boule, pour lesquels nous confirmons la conjecture de Shafarevich. Cette conjecture est un second fil directeur de cette thèse, que nous évoquerons plus loin dans cette introduction.

Quotients de la boule

Les sous-groupes discrets et sans torsion de $\mathrm{PU}(n,1)$ forment un sujet d'étude très riche. Bien que moins étudiés que leurs analogues contenus dans le groupe $\mathrm{PO}(n,1)$, ils ont néanmoins inspiré de nombreux travaux. Nous renvoyons à [Kap22] pour un article de survol sur ces objets. Il existe une taxinomie de ces sous-groupes qui commence par distinguer entre les groupes dits *élémentaires* et *non élémentaires*.

Les sous-groupes élémentaires et sans torsion de $\mathrm{PU}(n,1)$ sont de deux types : ils peuvent être *loxodromiques* ou *paraboliques*. Il y a peu de sous-groupes loxodromiques : ils sont virtuellement cycliques et donnent tous des quotients Stein (voir l'exemple 1 de la Section 2.1.1 ou [Che13, Theorem 1.1]). Les sous-groupes paraboliques sont plus intéressants et s'identifient aux sous-groupes discrets de $\mathrm{U}(n-1) \ltimes N$, où N est le groupe de Heisenberg de dimension réelle $2n-1$ (voir Section 1.2.3). Un sous-groupe parabolique unipotent discret de $\mathrm{PU}(n,1)$ s'identifie à un sous-groupe discret de N. On voit facilement qu'un quotient de $\mathbb{H}^n_{\mathbb{C}}$ par un sous-groupe parabolique est Stein si et seulement s'il est holomorphiquement convexe. Miebach a obtenu le résultat suivant, qui est une caractérisation complète des sous-groupes paraboliques unipotents discrets Γ de $\mathrm{PU}(n,1)$ pour lesquels $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein. Dans cet énoncé, une sous-algèbre de Lie nilpotente $\mathfrak{n}$ de l'algèbre de Lie de G s'identifie naturellement à un sous-espace de l'espace tangent à $\mathbb{H}^n_{\mathbb{C}}$ en un point o, et elle est dite totalement réelle si $\mathfrak{n} \cap J\mathfrak{n} = \{0\}$, où J est la structure complexe sur $T_o\mathbb{H}^n_{\mathbb{C}}$.

Théorème ([Mie24, Theorem 1.4])**.** Soit Γ un sous-groupe parabolique unipotent discret de $\mathrm{PU}(n,1)$, N_Γ son adhérence de Zariski, et $\mathfrak{n}_\Gamma$ l'algèbre de Lie de N_Γ. Alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein si et seulement si $\mathfrak{n}_\Gamma$ est totalement réelle.

Nous avons étendu cette caractérisation aux sous-groupes paraboliques (voir Théorème 2.1) ce qui permet d'obtenir une réponse complète à notre problème dans le cas des sous-groupes élémentaires et sans-torsion de $\mathrm{PU}(n,1)$. Préalablement aux travaux de Miebach, des résultats partiels avaient été obtenus dans les articles [dFab98 ; dFI01 ; Mie10 ; Che13]. On notera qu'un sous-groupe discret virtuellement abélien et sans torsion Γ de $\mathrm{PU}(n,1)$ est nécessairement élémentaire, et une conséquence de nos résultats est que si $\Gamma < \mathrm{PU}(n,1)$ est discret, virtuellement abélien et sans-torsion, alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein (voir Corollaire 2.8).

Parmi les sous-groupes non élémentaires, on distingue deux familles emboîtées l'une dans l'autre : les groupes *convexes-cocompacts* et les groupes *géométriquement finis*, que nous définissons dans les Sections 1.2.5 et 1.2.6, respectivement. Pour les sous-groupes convexes-cocompacts, la question de la convexité holomorphe des quotients est résolue : Yue [Yue99] puis Dey et Kapovich [DK20] ont montré que le quotient de l'espace hyperbolique complexe $\mathbb{H}^n_{\mathbb{C}}$ par un sous-groupe convexe-cocompact de $\mathrm{PU}(n,1)$ est toujours holomorphiquement convexe. Dans cette thèse, nous résolvons cette question pour les sous-groupes géométriquement finis. Le quotient de $\mathbb{H}^n_{\mathbb{C}}$ par un sous-groupe géométriquement fini Γ de $\mathrm{PU}(n,1)$ n'est pas nécessairement holomorphiquement convexe. Nous obtenons une caractérisation complète des quotients géométriquement finis vérifiant cette propriété, que nous décrivons maintenant, en renvoyant à la Proposition 2.10 pour des résultats plus complets. Le quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ a un nombre fini de pointes paraboliques, correspondant à des classes de conjugaison de sous-groupes paraboliques maximaux de Γ. Alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est holomorphiquement convexe si et seulement si pour pour tout sous-groupe parabolique maximal P de Γ, le quotient $\mathbb{H}^n_{\mathbb{C}}/P$ est holomorphiquement convexe (ou, de manière équivalente, Stein). La question de l'absence de sous-ensemble analytique compact de dimension positive est très riche ; des critères existent qui sont valables pour tous les sous-groupes discrets et sans torsion de $\mathrm{PU}(n,1)$. Ils seront décrits dans la suite.

Avant cela, et pour illustrer la richesse de cette dernière question, considérons une représentation fidèle et discrète $\rho : \Gamma \to \mathrm{PU}(n,1)$, où Γ est le groupe fondamental d'une surface compacte et sans bord de genre $g \geq 2$. On peut alors associer à ρ un nombre $\tau(\rho)$ appelé *invariant de Toledo*, qui ne peut prendre qu'un nombre fini de valeurs comprises entre $2-2g$ et $2g-2$. Nous renvoyons à l'exemple 5 de la section 1.2.5 pour plus de détails et des références. Si ρ est une *représentation maximale*, au sens où $|\tau(\rho)| = 2g-2$, alors le quotient $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ est holomorphiquement convexe et contient une sous-variété complexe compacte de dimension 1 : ce n'est donc pas une variété de Stein. Si $\tau(\rho) = 0$, alors $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ ne contient pas de sous-ensemble analytique compact de dimension positive. Bronstein a construit des exemples de représentations convexes-cocompactes ρ avec $\tau(\rho)$ compris strictement entre 0 et $2g-2$, telles que le quotient $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ contient une sous-variété complexe compacte : il est donc holomorphiquement convexe mais pas de Stein [Bro25]. Pour ces représentations avec invariant de Toledo intermédiaire, il semblerait conjecturalement que la situation soit mixte, avec des quotients contenant des sous-ensembles analytiques compacts de dimension positive et d'autres n'en contenant pas – cependant, à part les exemples de Bronstein, nous ne connaissons pas d'exemple pour lesquels on sait dire si le quotient contient un sous-ensemble analytique compact. Notamment, l'article [GKL01] contient des exemples pour lesquels cette question est ouverte.

Au-delà des groupes géométriquement finis, nous disposons de très peu d'informations sur la convexité holomorphe du quotient (voir quand même la Proposition 2.2 pour un résultat dans cette généralité). Nous disposons par contre de critères permettant d'affirmer que le quotient ne contient pas de sous-ensemble analytique compact de dimension positive (voir la Proposition 2.7 pour une présentation exhaustive des critères connus). L'un de ces critères, obtenu par Dey et

Kapovich [DK20], porte sur l'exposant critique du groupe, qui est une mesure, de nature plutôt dynamique, de la taille des orbites. Plus précisément, l'exposant critique d'un sous-groupe discret Γ de $\mathrm{PU}(n,1)$ est le nombre réel défini par

$$\delta(\Gamma) = \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty\}, \tag{1}$$

où o est un point arbitraire de $\mathbb{H}^n_{\mathbb{C}}$ et d la distance hyperbolique complexe sur la boule, normalisée de sorte que la métrique riemannienne associée est à courbure sectionnelle comprise entre -4 et -1. Il ne dépend pas du choix de o. Nous renvoyons à la section 1.2.7 pour plus de détails sur cette notion. Dey et Kapovich ont montré que si $\delta(\Gamma) < 2$, alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ ne contient pas de sous-ensemble analytique compact de dimension positive. L'article [DK20] propose également les trois conjectures suivantes.

Conjecture 1 ([DK20, Conjecture 2])**.** Soit Γ un sous-groupe discret et sans torsion de $\mathrm{PU}(n,1)$. Si $\delta(\Gamma) < 2$, alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein.

Conjecture 2 ([DK20, Conjecture 17])**.** Soit Γ un sous-groupe convexe-cocompact et sans torsion de $\mathrm{PU}(n,1)$. Si $\delta(\Gamma) = 2$, alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ n'est pas de Stein si et seulement si Γ agit de manière cocompacte sur un disque complexe totalement géodésique contenu dans $\mathbb{H}^n_{\mathbb{C}}$.

Un groupe agissant de manière cocompacte sur un disque complexe totalement géodésique contenu dans $\mathbb{H}^n_{\mathbb{C}}$ sera appelé *groupe fuchsien complexe* dans la suite.

Conjecture 3 ([DK20, Conjecture 21])**.** Soit Γ un sous-groupe discret et sans torsion de $\mathrm{PU}(n,1)$. Si $\delta(\Gamma) < 2k$, alors $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ ne contient aucun sous-ensemble analytique compact de dimension supérieure ou égale à k.

Nous avons rappelé plus haut que Dey et Kapovich ont résolu la première conjecture dans le cas des groupes convexes-cocompacts. Voici l'état actuel de ces conjectures : la première reste ouverte, mais nous l'avons résolue dans le cas des sous-groupes paraboliques (Théorème B) et des sous-groupes géométriquement finis (Théorème A) de $\mathrm{PU}(n,1)$. Nous avons résolu la seconde (Théorème C), et la troisième a été résolue par Connell, McReynolds et Wang dans [CMW23, Théorème 1.5].

Parmi les questions ouvertes du domaine, on peut mentionner la Conjecture 1 de Dey et Kapovich en toute généralité, mais aussi une question initialement posée par [Che13] : soit Γ un sous-groupe discret et sans-torsion de $\mathrm{PO}(n,1)$ plongé dans $\mathrm{PU}(n,1)$ de manière à préserver une copie totalement géodésique de l'espace hyperbolique réel $\mathbb{H}^n_{\mathbb{R}}$ de dimension n. Le quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est-il une variété de Stein ? Il est facile de voir que dans ce cas $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ ne contient pas de sous-ensemble analytique compact de dimension positive, et donc $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein si et seulement si elle est holomorphiquement convexe. Les théorèmes A et B fournissent une réponse positive à cette question lorsque Γ est élémentaire ou géométriquement fini. La question suivante est à notre connaissance également ouverte : étant donnés deux éléments $g, h \in \mathrm{PU}(n,1)$ tels que le groupe Γ engendré par g et h est discret, peut-on énoncer des conditions suffisantes et « calculables » sur g et h permettant d'affirmer que $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein ? L'étude de certains exemples pourrait aussi être intéressante, à l'instar des quotients construits dans [GKL01] et dans [Gra15], qui sont convexe-cocompacts. Contiennent-ils des sous-ensembles analytiques compacts de dimension positive ? Enfin, on peut s'interroger sur ce qui se passe lorsqu'on déforme le groupe Γ. Si $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ ne contient pas de sous-ensemble analytique compact de dimension positive, resp. si $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est holomorphiquement convexe, est-ce qu'une petite déformation de Γ qui reste discrète aura la même propriété ?

Voici maintenant l'énoncé de nos résultats principaux sur les quotients de la boule.

Théorème A. *Soit* Γ *un sous-groupe géométriquement fini et sans torsion de* $\mathrm{PU}(n,1)$.

(a) Si Γ *est Gromov-hyperbolique, alors* $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ *est holomorphiquement convexe.*

(b) Si Γ *est un groupe libre ou si* $\delta(\Gamma) < 2$*, alors* $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ *est une variété de Stein.*

(c) Si Γ *préserve une sous-variété totalement réelle et totalement géodésique de* $\mathbb{H}^n_{\mathbb{C}}$*, alors* $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ *est une variété de Stein.*

Théorème B. *Soit* Γ *un sous-groupe parabolique discret et sans torsion de* $\mathrm{PU}(n,1)$.

(a) Si $\delta(\Gamma) < 2$ *ou si* Γ *préserve une sous-variété totalement réelle et totalement géodésique de* $\mathbb{H}^n_{\mathbb{C}}$*, alors* Γ *est virtuellement abélien.*

(b) Si Γ *est virtuellement abélien, alors* $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ *est une variété de Stein.*

Théorème C. *Soit* Γ *un sous-groupe discret et sans torsion de* $\mathrm{PU}(n,1)$ *tel que* $\delta(\Gamma) = 2$*. Supposons que* $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ *contienne un sous-ensemble analytique compact de dimension positive. Alors* Γ *est un groupe fuchsien complexe.*

La preuve du Théorème C utilise de façon cruciale la théorie de Patterson–Sullivan. Cette théorie a été introduite pour l'espace hyperbolique réel dans les articles [Pat76] et [Sul79], puis développée dans le contexte plus général des espaces $\mathrm{CAT}(-1)$ dans [Rob03], voir aussi [Cor90 ; CI99 ; Coo93]. Pour une variété riemannienne complète simplement connexe X à courbure inférieure ou égale à -1, elle consiste à associer à un groupe Γ agissant librement et proprement discontinument par isométries sur X une famille indexée par les points de X de mesures finies $(\mu_x)_{x\in X}$ sur le bord visuel de X, et qui se comporte bien sous l'action de Γ. Cette théorie a d'abord permis d'étudier certaines propriétés du flot géodésique sur X/Γ, puis elle a notamment été utilisée pour relier l'exposant critique de Γ et la dimension de Hausdorff d'un certain sous-ensemble du bord de X appelé ensemble limite conique de X [Cor90 ; Pau97]. Un des avantages de l'approche des conjectures de Dey et Kapovich par la théorie de Patterson–Sullivan est que celle-ci a été étendue pour des espaces symétriques $\mathbb{X} = G/K$ de rang supérieur, d'abord par Burger en rang 2 [Bur93] puis par Albuquerque [Alb99] et par Quint [Qui02a ; Qui02b]. Nous renvoyons également à [PS17] et [PSW23] pour des résumés de cette théorie.

Plus précisément, pour établir le Théorème C notre stratégie consiste à montrer que, sous des hypothèses appropriées, une certaine fonction construite dans [CMW23] à partir de la théorie de Patterson–Sullivan est (strictement) plurisousharmonique, puis à en déduire des conséquences géométriques. Cette preuve utilise que pour toute variété kählérienne complète simplement connexe à courbure sectionnelle inférieure ou égale à -1, les fonctions de Busemann B_ξ avec ξ dans le bord visuel de X vérifient la minoration

$$i\partial\bar{\partial}B_\xi \geq \omega, \tag{2}$$

et dans le cas de l'espace hyperbolique complexe à courbure comprise entre -4 et -1, il s'agit même d'une égalité. Nous renvoyons à la Section 1.2.4 pour la définition et les propriétés de ces fonctions de Busemann. Notons que la fonction définie dans [CMW23] à partir des mesures de Patterson–Sullivan a également été réutilisée pour étudier des propriétés d'expansion d'espaces localement symétriques dans [FL24].

En rang supérieur, notre approche consiste à imiter cette stratégie en remplaçant la notion d'exposant critique par l'entropie d'une certaine forme linéaire. Nous présentons dans la section suivante les résultats obtenus dans cette direction.

Quotients d'espaces symétriques hermitiens de rang supérieur (avec C. Davalo)

Nous nous tournons maintenant vers l'étude des quotients d'espaces symétriques hermitiens $\mathbb{X} = G/K$ de type non compact. En collaboration avec Colin Davalo, nous obtenons des résultats sur l'absence de sous-variétés analytiques compactes de dimension positive dans des quotients de la forme $\Gamma\backslash\mathbb{X}$[2], sous une certaine hypothèse sur le sous-groupe discret Γ de G. Cette hypothèse prend la forme d'une inégalité qui généralise celle proposée par Dey et Kapovich concernant l'exposant critique. En résumé, l'idée est de remplacer l'exposant critique par l'entropie associée à une forme linéaire correspondant au bord de Shilov de l'espace symétrique. Nous obtenons également une caractérisation du cas d'égalité (analogue au Théorème C), qui nous permet d'appliquer nos résultats à une large famille d'exemples.

Pour fixer les notations, soit G un groupe de Lie simple de type non compact. Dans cette thèse, cela signifie que G est un groupe de Lie connexe de centre fini, non Euclidien et d'algèbre de Lie simple. Soit $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ une décomposition de son algèbre de Lie $\mathfrak{g}$ induite par une involution de Cartan de $\mathfrak{g}$. Nous considérons $\mathbb{X} = G/K$ l'espace symétrique associé, obtenu en quotientant G par le sous-groupe compact maximal dont l'algèbre de Lie est $\mathfrak{k}$. Nous supposons que G est hermitien, ce qui signifie qu'il existe une structure de variété complexe (et même kählérienne) sur $\mathbb{X}$ pour laquelle les éléments de G agissent par isométries biholomorphes sur $\mathbb{X}$. Via l'application exponentielle, les sous-algèbres abéliennes maximales $\mathfrak{a}$ de $\mathfrak{p}$ correspondent aux plats maximaux de $\mathbb{X} = G/K$ passant par le point $eK \in \mathbb{X}$. Nous fixons une telle sous-algèbre $\mathfrak{a}$ ainsi qu'une chambre de Weyl $\mathfrak{a}^+ \subset \mathfrak{a}$. La décomposition de Cartan $G = K \exp \mathfrak{a}^+ K$ de G permet alors de construire une projection de Cartan $\mu\colon G \to \mathfrak{a}^+$. Toutes ces notions classiques seront rappelées plus en détail avec des références dans la Section 1.3.

Dans le bord visuel $\partial_{\mathrm{vis}}\mathbb{X}$ de $\mathbb{X}$, il existe une G-orbite compacte appelée le *bord de Shilov* de $\mathbb{X}$. Cette G-orbite intersecte le bord du cône $\exp(\mathfrak{a}^+)$ en exactement un point ξ, et il existe une unique racine simple $\breve{a} \in \mathfrak{a}^*$ qui ne s'annule pas sur la demi-droite ouverte allant de l'origine à ce point. Cette racine sera appelée racine associée au bord de Shilov de $\mathbb{X}$. Nous renvoyons à la Définition 1.44 pour plus de détails.

L'hypothèse généralisant celle de Dey et Kapovich en rang supérieur fait intervenir l'entropie d'un sous-groupe discret associée à cette racine simple $\breve{a}$. L'entropie associée à une forme linéaire $\phi \in \mathfrak{a}^*$ a été introduite dans [Bur93 ; Alb99 ; Qui02a ; Qui02b]. Contrairement à la section précédente, nous préférons ici travailler avec une représentation fidèle et discrète $\rho\colon \Gamma \to G$ d'un groupe de type fini Γ dans G, plutôt qu'avec un sous-groupe discret de G de type fini. L'entropie de ρ associée à la racine $\breve{a}$ est le nombre défini par

$$h_\rho(\breve{a}) := \inf\left\{ s \in \mathbb{R} \;\middle|\; \sum_{\gamma\in\Gamma} e^{-s\breve{a}(\mu(\rho(\gamma)))} < +\infty \right\} \in \mathbb{R} \cup \{+\infty\}.$$

Par ailleurs, remarquons que la Formule (1) permet de définir l'exposant critique du groupe Γ dans ce contexte. Nous pouvons maintenant énoncer notre résultat principal en rang supérieur, obtenu en collaboration avec Colin Davalo.

[2]Comme nous écrivons $\mathbb{X} = G/K$ pour l'espace symétrique, nous quotientons à gauche et non à droite par Γ lorsque nous travaillons en rang supérieur.

Théorème D. *Soit $\rho\colon \Gamma \to G$ une représentation discrète et fidèle d'un groupe sans torsion Γ dans un groupe de Lie classique, simple et hermitien G de type non compact. On suppose que l'exposant critique $\delta(\rho(\Gamma))$ de $\rho(\Gamma)$ est non nul. Fixons un plat maximal $\exp(\mathfrak{a})$ dans l'espace symétrique $\mathbb{X}$ associé à G, avec $\mathfrak{a} \subset \mathfrak{p}$, soit $\breve{a} \in \mathfrak{a}^*$ la racine associée au bord de Shilov de $\mathbb{X}$, et soit $h_\rho(\breve{a})$ l'entropie de ρ associée à la racine $\breve{a}$.*

1. *Si $h_\rho(\breve{a}) < 1$, alors $\rho(\Gamma)\backslash\mathbb{X}$ ne contient aucune sous-variété analytique compacte de dimension positive.*
2. *Si $h_\rho(\breve{a}) = 1$ et si $\rho(\Gamma)$ est non élémentaire, alors $\rho(\Gamma)\backslash\mathbb{X}$ contient une sous-variété analytique compacte de dimension positive si et seulement si Γ est le groupe fondamental d'une surface et $\rho(\Gamma)$ préserve un disque complexe totalement géodésique dans $\mathbb{X}$.*

La signification de « non élémentaire » sera donnée dans la Section 1.3.5 et la racine associée au bord de Shilov de $\mathbb{X}$ est définie dans la Définition 1.44. En fait, on peut renforcer la conclusion du Théorème D-2 : le disque complexe totalement géodésique préservé par $\rho(\Gamma)$ est un *disque diagonal*, c'est-à-dire l'image de la diagonale par un plongement totalement géodésique d'un polydisque maximal (voir la Définition 2.13).

Soit S_g une surface de genre $g \geq 2$. Il a été montré dans [PSW23, Theorem 9.8] que $h_\rho(\breve{a}) = 1$ pour toute représentation maximale ρ du groupe fondamental $\pi_1(S_g)$ de S_g. De cela, nous déduisons le corollaire suivant.

Corollaire E. *Soit G un groupe de Lie simple hermitien de type non compact et $\mathbb{X}$ son espace symétrique associé. Soit $\rho\colon \pi_1(S_g) \to G$ une représentation maximale. Alors $\rho(\Gamma)\backslash\mathbb{X}$ contient une sous-variété analytique compacte de dimension positive si et seulement si $\rho(\Gamma)$ préserve un disque diagonal dans $\mathbb{X}$.*

Comme expliqué précédemment, la stratégie utilisée pour démontrer le Théorème D est une élaboration des arguments utilisés pour démontrer le Théorème C. Dans le cadre des variétés de Cartan–Hadamard, les fonctions de Busemann sont convexes (voir [EO73]), et par conséquent, pour un espace symétrique hermitien de type non compact, les fonctions de Busemann sont plurisousharmoniques (nous renvoyons à la Section 1.1.1 pour le lien entre convexité et plurisousharmonicité en géométrie kählérienne). En général, elles ne sont pas strictement plurisousharmoniques et ne satisfont pas l'inégalité (2). Une étape d'intérêt indépendant dans la démonstration du théorème D est une borne inférieure de la forme de Levi des fonctions de Busemann en rang supérieur, qui améliore [Eys97, Proposition 4.4.4] et repose sur le calcul du hessien des fonctions de Busemann pour les espaces symétriques effectué dans [Dav23].

Proposition F. *Soit G un groupe de Lie simple hermitien de type non compact et $(\mathbb{X}, \omega)$ son espace symétrique associé vu comme une variété kählérienne. Fixons un plat maximal $\exp(\mathfrak{a})$ de $\mathbb{X}$ avec $\mathfrak{a} \subset \mathfrak{g}$ et une chambre de Weyl $\mathfrak{a}^+ \subset \mathfrak{a}$, et soit $\breve{a} \in \mathfrak{a}^*$ la racine associée au bord de Shilov de $\mathbb{X}$.*

Soit $w \in \mathfrak{a}^+$ un vecteur unitaire, et soit ξ_w le point du bord visuel de $\mathbb{X}$ associé à w. Choisissons une fonction de Busemann $B_{\xi_w} : \mathbb{X} \to \mathbb{R}$ au point ξ_w. Deux telles fonctions ne diffèrent que d'une constante additive, de sorte que leur forme de Levi est indépendante de ce choix. Elle satisfait l'inégalité :

$$i\partial\bar{\partial}B_{\xi_w} \geq \frac{\breve{a}(w)}{2}\omega.$$

De plus, l'égalité a lieu si et seulement si ξ_w appartient au bord de Shilov de $\mathbb{X}$.

En particulier, la fonction de Busemann B_{ξ_w} est strictement plurisousharmonique dès lors que $\breve{a}(w) \neq 0$, c'est-à-dire en dehors d'une face du simplexe obtenu par projectivisation de la chambre de Weyl, qui est la face opposée au sommet correspondant au bord de Shilov de $\mathbb{X}$. Pour un point ξ_w appartenant à cette face, la fonction de Busemann B_{ξ_w} n'est pas strictement plurisousharmonique, comme on le verra dans la preuve de la Proposition F.

La forme kählérienne ω induite par la structure hermitienne sur $\mathbb{X}$ n'est définie qu'à un multiple scalaire positif près. On peut vérifier que l'inégalité obtenue dans la proposition est homogène par changement d'échelle de la forme kählérienne. En cohérence avec l'inégalité (2), on peut vérifier que la métrique satisfaisant $\breve{a}(\breve{w}) = 2$ est l'unique métrique sur $\mathbb{X}$ telle que :

$$\sup_{v \in T\mathbb{X}} \mathrm{HSC}(v) = -4,$$

où HSC désigne la courbure sectionnelle holomorphe de $\mathbb{X}$, voir [CØ03].

Compactifications toroïdales de quotients de la boule

Nous décrivons maintenant un autre aspect de nos travaux, dont le fil directeur est le suivant :

> Étant donnée une variété complexe X qui n'est pas de Stein (typiquement une variété compacte), montrer sous des hypothèses adéquates que le revêtement universel de X est une variété de Stein.

En dimension 1, le théorème d'uniformisation implique que le revêtement universel d'une surface de Riemann compacte est soit compact, soit une variété de Stein. Dans les deux cas, c'est une variété holomorphiquement convexe. La conjecture de Shafarevich, formulée en 1972 pour les variétés projectives, propose d'étendre ce constat en dimension supérieure :

Conjecture 4 (Shafarevich, [Sha13])**.** Le revêtement universel de toute variété kählérienne compacte est holomorphiquement convexe.

Cette conjecture est ouverte en général, et on ignore même s'il existe toujours une fonction holomorphe non constante sur le revêtement universel d'une variété kählérienne compacte à groupe fondamental infini. Elle est néanmoins connue dans le cas où le groupe fondamental de la variété est linéaire [Eys+12 ; CCE15], ainsi que pour certains exemples spécifiques [EM15]. Voir aussi [Kat97 ; KR98 ; Eys04] pour des résultats antérieurs, ainsi que [Eys11] pour un article de synthèse sur cette conjecture. Enfin, comme nous l'avons déjà remarqué, elle est connue dans le cas d'une variété kählérienne compacte à courbure sectionnelle négative ou nulle puisque le revêtement universel d'une telle variété est de Stein.

Il existe une famille de variétés kählériennes, appelées *compactifications toroïdales de quotients de la boule*, ou parfois *compactifications de Mumford*, qui ont été décrites pour la première fois dans [Mum75 ; Mok12 ; Ash+10] et qui sont obtenues en compactifiant le quotient de la boule $\mathbb{H}^n_{\mathbb{C}}$ par un réseau non uniforme et sans-torsion Γ dont les sous-groupes paraboliques maximaux sont unipotents. Pour tout réseau non uniforme Γ_0 de $\mathrm{PU}(n, 1)$, il existe un sous-groupe d'indice fini $\Gamma < \Gamma_0$ vérifiant cette hypothèse, d'après [Hum98]. Nous noterons dans la suite X_Γ la compactification toroïdale de $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. Elle est réalisée en « recollant » un tore de dimension complexe $n-1$ au bout de chaque pointe parabolique de $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, et sa construction est décrite en détail dans la Section 3.1. Les compactifications toroïdales sont en fait des variétés projectives

[Mok12 ; DD15], et plusieurs auteurs ont étudié leurs propriétés du point de vue de la géométrie algébrique [DD17 ; BT18 ; Cad21 ; Mem23], ou sous d'autres angles [Py17 ; DS18]. On ignore si le groupe fondamental des compactifications toroïdales est linéaire.

Il existe trois types de résultats portant sur ces variétés. Certains sont vrais pour toutes les variétés X_Γ. D'autres sont vraies à partir d'une certaine dimension seulement (par exemple si $n \geq 6$, alors le fibré canonique de X_Γ est ample, ce qui est connu pour être faux pour certaines compactifications toroïdales en dimension 2, voir [BT18]). Enfin, certains résultats montrent que pour tout réseau non uniforme Γ_0 de $\mathrm{PU}(n,1)$, il existe un sous-groupe Γ' d'indice fini dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe Γ'' de Γ' d'indice fini, une certaine propriété P est vraie sur la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$. Par exemple, le résultat suivant de Hummel et Schroeder concerne l'étude métrique de ces variétés, et a une grande importance pour nos travaux.

Théorème ([HS96, Théorèmes 2 et 7])**.** Soit Γ un réseau non uniforme et sans torsion de $\mathrm{PU}(n,1)$ dont les sous-groupes paraboliques sont unipotents. Alors X_Γ est une variété kählérienne. De plus, il existe un sous-groupe d'indice fini $\Gamma' < \Gamma$ tel que, pour tout sous-groupe d'indice fini $\Gamma'' < \Gamma'$, la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ admet une métrique riemannienne à courbure sectionnelle négative ou nulle.

Savoir si les compactifications toroïdales vérifient la conjecture de Shafarevich est encore une question ouverte. Une question plus abordable est probablement la suivante : Soit Γ_0 un réseau non uniforme de $\mathrm{PU}(n,1)$. Existe-t-il un sous-groupe Γ' d'indice fini de Γ_0 dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe Γ'' de Γ' d'indice fini, la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ vérifie la conjecture de Shafarevich ? Nous obtenons une réponse positive à cette seconde question, sous l'hypothèse supplémentaire que Γ_0 est un réseau arithmétique. On peut déjà remarquer que, quitte à remplacer Γ' par un sous-groupe Γ'' d'indice fini, cette question équivaut à demander si le revêtement universel de $X_{\Gamma''}$ est Stein. Ceci découle en effet du résultat déjà cité de Hummel et Schroeder qui implique qu'on peut trouver un sous-groupe d'indice fini Γ' de Γ_0 tel que pour tout sous-groupe d'indice fini Γ'' de Γ', le revêtement universel de $X_{\Gamma''}$ est contractile et en particulier ne contient pas de sous-ensemble analytique compact de dimension positive.

Préalablement à cela, et de manière indépendante, nous obtenons des résultats qui complètent et précisent le théorème de Hummel et Schroeder. Nous montrons en effet que les compactifications toroïdales X_Γ n'admettent aucune métrique kählérienne à courbure sectionnelle négative ou nulle (Théorème G), confirmant ainsi une affirmation de [HS96], mais qu'elles admettent une métrique kählérienne à courbure bisectionnelle holomorphe négative ou nulle (Théorème H). Ceci nous permet de répondre positivement à une question posée par Diverio dans [Div22] sur l'existence d'une variété complexe compacte n'admettant aucune métrique kählérienne à courbure sectionnelle holomorphe strictement négative mais admettant une métrique kählérienne à courbure (bi)sectionnelle holomorphe *quasi-négative* au sens de [DT19, Definition 1.1], c'est-à-dire que la courbure bisectionnelle holomorphe est partout négative ou nulle et négative en un point (Corollaire I).

Nous montrons ensuite que le revêtement universel $\widetilde{X_\Gamma}$ de X_Γ est une variété de Stein lorsque le réseau Γ est arithmétique et « suffisamment profond ». Pour ce faire, nous étudions les propriétés de l'application d'Albanese des variétés obtenues par compactification de revêtements finis de $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, et nous obtenons des résultats sur cette application en restriction aux tores ajoutés lors de la compactification (Théorème J), prolongeant un travail antérieur d'Eyssidieux [Eys18].

Cette amélioration nous permet de démontrer que pour tout réseau arithmétique non uniforme et sans-torsion Γ_0, il existe un sous-groupe d'indice fini $\Gamma' < \Gamma_0$ dont les sous-groupes paraboliques sont unipotents et tels que pour tout sous-groupe d'indice fini $\Gamma'' \subset \Gamma'$, le revêtement universel de la compactification toroïdale de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ est une variété de Stein (Théorème K). Lorsque $n = 2$, ce résultat avait été obtenu auparavant par Eyssidieux [Eys18].

Décrivons maintenant plus précisément ces résultats. Dans [HS96], Hummel et Schroeder affirment que les compactifications toroïdales n'admettent pas de métrique kählérienne à courbure sectionnelle négative ou nulle. Une démonstration de cette assertion a été donnée en dimension complexe 2 pour des réseaux « suffisamment profonds » par Di Cerbo dans [Di 12, Théorème A]. En généralisant cette approche en toute dimension, nous démontrons le résultat suivant.

Théorème G. *Soit Γ un réseau non uniforme et sans torsion de $\mathrm{PU}(n, 1)$ dont les sous-groupes paraboliques sont unipotents. Alors la compactification toroïdale X_Γ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ n'admet aucune métrique kählérienne à courbure sectionnelle négative ou nulle.*

En particulier, les techniques de construction de métriques kählériennes sur X_Γ développées dans [HS96] ne permettent pas de produire une métrique à courbure sectionnelle négative ou nulle. Cependant, nous montrons par un calcul qu'elles sont suffisantes pour construire une métrique kählérienne dont la courbure vérifie une condition plus faible de non-positivité, à savoir la non-positivité de la courbure bisectionnelle holomorphe.

Théorème H. *Soit Γ_0 un réseau non uniforme et sans torsion de $\mathrm{PU}(n, 1)$. Il existe un sous-groupe d'indice fini $\Gamma' < \Gamma_0$ dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe d'indice fini $\Gamma'' < \Gamma'$, la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ admet une métrique kählérienne à courbure bisectionnelle holomorphe négative ou nulle.*

La métrique construite dans la démonstration de ce théorème est à courbure bisectionnelle holomorphe négative sur le sous-ensemble ouvert $\mathbb{H}^n_{\mathbb{C}}/\Gamma'' \subset X_{\Gamma''}$. En particulier, elle est quasi-négative. Dans leur article, Diverio et Trapani ont montré que toute variété kählérienne compacte à courbure sectionnelle holomorphe quasi-négative possède un fibré canonique ample, un résultat déjà connu sous l'hypothèse de négativité de la courbure sectionnelle holomorphe [WY16, Theorem 2], [TY17, Corollary 1.3]. Cela répondait à une question posée par Yau. Cette situation a naturellement conduit Diverio à se demander dans [Div22] s'il existe des variétés kählériennes compactes à courbure sectionnelle holomorphe quasi-négative qui n'admettent pas de métrique kählérienne à courbure sectionnelle holomorphe négative. Les compactifications toroïdales contiennent des tores, qui ne sont pas hyperboliques au sens de Kobayashi, et permettent ainsi de répondre positivement à cette question.

Corollaire I. *Pour Γ'' un réseau non uniforme comme dans le Théorème H, la variété complexe $X_{\Gamma''}$ est un exemple de variété compacte qui admet une métrique kählérienne à courbure bisectionnelle holomorphe quasi-négative, mais qui n'admet pas de métrique kählérienne à courbure sectionnelle holomorphe strictement négative.*

Au passage, on observe que le Théorème H, combiné au résultat principal de [DT19], fournit une démonstration alternative de [DD17, Théorème 1.3] : pour un réseau non uniforme Γ'' comme dans le Théorème H, le fibré canonique de X''_Γ est ample. Pour une démonstration plus directe, nous remarquons dans la Section 3.3 que la métrique construite dans la preuve du Théorème H est à courbure de Ricci négative. De plus, en appliquant [Gue22, Théorème 3.1], on obtient que toute sous-variété irréductible de $X_{\Gamma''}$ qui n'est pas contenue dans le complémentaire de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ est de type général.

Nous énonçons maintenant nos résultats en lien avec la conjecture de Shafarevich pour les compactifications toroïdales X_Γ. Soit Γ_0 un réseau arithmétique non uniforme et sans-torsion de $\mathrm{PU}(n,1)$. Eyssidieux a montré qu'il existe un sous-groupe d'indice fini $\Gamma' < \Gamma_0$ dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe d'indice fini $\Gamma'' < \Gamma'$, l'application d'Albanese $A_{\Gamma''}\colon X_{\Gamma''} \longrightarrow \mathrm{Alb}(X_{\Gamma''})$ de la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ est une immersion sur le complémentaire des tores ajoutés lors de la compactification [Eys18, Théorème 1.3]. Il en a alors déduit en dimension 2 que le revêtement universel de $X_{\Gamma''}$ est holomorphiquement convexe, en utilisant un résultat de [Nap90] valable uniquement dans cette dimension.

Pour aborder le cas de la dimension supérieure, nous précisons le résultat d'Eyssidieux sur les applications d'Albanese. Pour cela, nous analysons le comportement de l'application d'Albanese sur les tores ajoutés lors de la compactification, que nous appellerons dans la suite « tores à l'infini de X_Γ ». Plus précisément, un tore à l'infini de X_Γ est une composante connexe du complémentaire de $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ dans X_Γ. Avec cette terminologie, nous démontrons le résultat suivant.

Théorème J. *Soit Γ_0 un réseau arithmétique non uniforme et sans torsion de $\mathrm{PU}(n,1)$. Il existe un sous-groupe d'indice fini $\Gamma' < \Gamma_0$ dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe d'indice fini $\Gamma'' < \Gamma'$, la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ vérifie les propriétés suivantes :*

- *Son application d'Albanese $A_{\Gamma''}$ est une immersion sur l'ouvert $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X''_\Gamma$.*
- *Pour tout tore à l'infini T de $X_{\Gamma''}$, l'application $A_{\Gamma''}|_T$ est soit une immersion, soit constante.*

L'arithméticité de Γ_0 est utilisée de manière cruciale dans la preuve du Théorème J, comme dans [Eys18]. Déterminer si, pour tout tore à l'infini T de X_{Γ_0}, il existe un sous-groupe d'indice fini $\Gamma < \Gamma_0$ pour lequel l'application d'Albanese A_Γ de X_Γ est une immersion lorsqu'on la restreint aux tores au-dessus de T, semble bien plus difficile. Cependant, le résultat plus faible énoncé dans le Théorème J suffit à nos objectifs. Nous le combinons aux arguments de [Nap90], valables en toute dimension, concernant la convexité holomorphe par rapport à un fibré en droites suffisamment positif, et nous adaptons les raisonnements donnés dans cet article en dimension 2 afin de généraliser le résultat d'Eyssidieux en toute dimension :

Théorème K. *Soit Γ_0 un réseau arithmétique non uniforme et sans torsion de $\mathrm{PU}(n,1)$. Il existe un sous-groupe d'indice fini $\Gamma' < \Gamma_0$ dont les sous-groupes paraboliques sont unipotents et tel que pour tout sous-groupe d'indice fini $\Gamma'' < \Gamma'$, le revêtement universel $\widetilde{X''_\Gamma}$ de la compactification toroïdale $X_{\Gamma''}$ de $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ est une variété de Stein.*

Ce théorème se déduit de la Proposition 3.15 énoncée dans la Section 3.5, qui est un résultat plus général concernant des variétés projectives qui sont le domaine d'une application holomorphe génériquement finie à valeurs dans une variété compacte dont le revêtement universel est de Stein.

Résumé des chapitres

Cette thèse se compose de trois chapitres.

- Le premier décrit les notions préliminaires permettant de prouver les résultats des deux premières sections de cette introduction. Dans la première section, nous commençons par rappeler les notions d'analyse complexe utilisées (plurisousharmonicité, sous-ensembles analytiques, convexité holomorphe, variétés de Stein). Ensuite, dans la Section 1.2, nous décrivons en détail l'espace hyperbolique complexe et son groupe de biholomorphismes $\mathrm{PU}(n,1)$. Nous introduisons les notions d'ensemble limite et d'ensemble de discontinuité, et discutons de la structure du stabilisateur d'un point à l'infini de l'espace hyperbolique complexe. Nous définissons les sous-groupes convexes-cocompacts et les sous-groupes géométriquement finis de $\mathrm{PU}(n,1)$. Nous introduisons la notion d'exposant critique, et nous décrivons brièvement la théorie de Patterson–Sullivan en rang 1. Enfin, dans la Section 1.3 nous décrivons quelques résultats de structure des espaces symétriques hermitiens de rang supérieur, dont nous aurons besoin dans la suite : il s'agit de la décomposition en racines d'un groupe de Lie simple de type noncompact, des notions de chambres de Weyl et de projection de Cartan du groupe, du bord visuel de l'espace symétrique associé, des fonctions de Busemann en rang supérieur et de la théorie de Patterson–Sullivan en rang supérieur, développée notamment par Quint dans [Qui02a; Qui02b] via les notions de fonction indicatrice de croissance et d'entropie.

- Le deuxième chapitre contient, dans la Section 2.1, les preuves des résultats portant sur les quotients de la boule. Nous commençons par énoncer des résultats valables dans le contexte plus général des variétés kählériennes complètes simplement connexes à courbure sectionnelle inférieure ou égale à -1, et nous présentons des critères permettant d'affirmer que le quotient d'une telle variété ne contient pas de sous-ensemble analytique compact de dimension positive (Proposition 2.7). Ensuite, nous donnons une caractérisation complète des sous-groupes paraboliques discrets Γ de $\mathrm{PU}(n,1)$ pour lesquels $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est une variété de Stein (Théorème 2.1), nous prouvons le Théorème B et nous donnons des exemples de quotients paraboliques de la boule. Passant aux quotients géométriquement finis, nous énonçons et prouvons la caractérisation mentionnée plus haut des sous-groupes géométriquement finis Γ pour lesquels $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ est holomorphiquement convexe (Proposition 2.10), puis nous prouvons le Théorème A. Dans la Section 2.2, nous présentons les preuves des résultats obtenus en collaboration avec C. Davalo en rang supérieur (Théorèmes D et Proposition F).

- Le troisième chapitre concerne les compactifications toroïdales de quotients de la boule. Dans la Section 3.1, nous rappelons leur construction en commençant par décrire le quotient de la boule par un élément non trivial dans le centre du radical unipotent de $\mathrm{PU}(n,1)$. Dans la section 3.2, nous montrons un résultat général (Proposition 3.4) qui implique le théorème G. Ensuite, nous construisons dans la section 3.3 une métrique kählérienne à courbure bisectionnelle holomorphe quasi-négative et à courbure de Ricci négative sur les compactifications toroïdales des quotients de la boule par un réseau « assez profond » (Théorème H). Nous passons ensuite à nos résultats concernant la conjecture de Shafarevich pour les compactifications toroïdales dans les sections 3.4 et 3.5 qui contiennent respectivement les preuves des Théorèmes J et K.

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Introduction in English

This thesis focuses on the existence of holomorphic functions on certain complex manifolds. An open set $X \subset \mathbb{C}^n$ admits, via restrictions of functions defined on $\mathbb{C}^n$, sufficiently many holomorphic functions $f\colon X \to \mathbb{C}$ so that the following separability property is satisfied:

$$\forall x \neq y \in X, \exists f\colon X \to \mathbb{C} \text{ holomorphic such that } f(x) \neq f(y). \tag{S}$$

However, not all open subsets of $\mathbb{C}^n$ satisfy the following property:

$$\forall (x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}} \text{ without accumulation point}, \exists f\colon X \to \mathbb{C} \text{ holomorphic s.t. } \sup_n |f(x_n)| = +\infty. \tag{HC}$$

For example, a punctured ball in $\mathbb{C}^2$ does not satisfy this property, since any holomorphic function defined on a punctured ball of dimension at least 2 extends holomorphically to 0, see for instance [Kra01, Theorem 1.2.6].

More generally, a complex manifold X is called *holomorphically convex* if it satisfies property (**HC**). If, in addition, the holomorphic functions on X *separate points*, in the sense of property (**S**), we say that X is a *Stein manifold*. Stein manifolds admit many equivalent definitions, as we will see in Section 1.1.4. Throughout this work, we will use the following characterization, which is a consequence of the existence of Remmert's reduction: a manifold X is Stein if and only if it is holomorphically convex and does not contain any compact analytic subvariety of positive dimension (see Definition 1.15 and the references in Section 1.1.4).

A compact manifold of positive dimension never satisfies (**S**) and trivially satisfies (**HC**), but for noncompact manifolds the properties (**S**) and (**HC**) are independent and may or may not hold. A covering space (possibly branched) of a Stein manifold is Stein [Le 76, Théorème 1], but the quotient of a Stein manifold is not generally Stein. One of the guiding threads of this thesis is the following:

> Given a simply connected complex manifold X, obtain criteria on the groups Γ acting freely and properly discontinuously by biholomorphisms on X that allow one to assert that X/Γ is a Stein manifold.

For at least one quotient X/Γ to be Stein, it is necessary that X be a Stein manifold, according to the result recalled above. A result of Greene and Wu ([GW79, Proposition 1.17], see also the Example in Section 1.1.4) states that a complete, simply connected Kähler manifold with nonpositive sectional curvature is a Stein manifold. We will restrict our study to quotients of such manifolds, and add the assumption that the groups act by biholomorphic *isometries* on these manifolds. In this context, we further restrict our study to two types of complete, simply connected Kähler manifolds: on the one hand, those with sectional curvature less than or equal to -1, and on the other hand, Hermitian symmetric spaces of noncompact type.

The intersection of these two types of manifolds yields exactly the family of *complex hyperbolic spaces* $\mathbb{H}^n_{\mathbb{C}}$, which are manifolds biholomorphic to the unit ball in $\mathbb{C}^n$, equipped with a Kähler metric whose sectional curvature varies between -4 and -1. It is for these spaces that we obtain our main results. Let us note immediately that every biholomorphism of the unit ball is an isometry of complex hyperbolic space, and that the group of biholomorphisms of $\mathbb{H}^n_{\mathbb{C}}$ is the group $\mathrm{PU}(n,1)$, defined in Section 1.2.1. Moreover, a subgroup Γ of $\mathrm{PU}(n,1)$ acts freely and properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$ if and only if it is discrete and torsion-free. In the case of the unit ball in $\mathbb{C}^n$, this work is motivated by questions raised by Dey and Kapovich in [DK20], which we will discuss in more detail later.

In this introduction, we begin by describing the results obtained on the quotients of the ball by discrete, torsion-free subgroups of $\mathrm{PU}(n,1)$, which were the subject of the preprint [Sar24]. Some of our proof techniques use Patterson–Sullivan theory and extend to higher rank. In the second part of the introduction, we describe unpublished results concerning quotients of higher rank Hermitian symmetric spaces, obtained in collaboration with Colin Davalo. Finally, a third section concerns the results from the preprint [Sar23] on the universal cover of certain toroidal compactifications of ball quotients, for which we confirm the Shafarevich conjecture. This conjecture is a second guiding thread of this thesis, which we will address later in the introduction.

Quotients of the ball

The discrete and torsion-free subgroups of $\mathrm{PU}(n,1)$ form a very rich field of study. Although less explored than their analogues within the group $\mathrm{PO}(n,1)$, they have nevertheless inspired a great deal of research. We refer to [Kap22] for a general survey on these objects. There exists a taxonomy of these subgroups, beginning with the distinction between so-called *elementary* and *nonelementary* groups.

The elementary and torsion-free subgroups of $\mathrm{PU}(n,1)$ come in two types: they can be *loxodromic* or *parabolic*. There are few loxodromic subgroups: they are virtually cyclic and all yield Stein quotients (see Example 1 in Section 2.1.1 or [Che13, Theorem 1.1]). Parabolic subgroups are more interesting and can be identified with discrete subgroups of $\mathrm{U}(n-1) \ltimes N$, where N is the Heisenberg group of real dimension $2n-1$ (see Section 1.2.3). A discrete unipotent parabolic subgroup of $\mathrm{PU}(n,1)$ corresponds to a discrete subgroup of N. One can easily see that a quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a parabolic subgroup is Stein if and only if it is holomorphically convex. Miebach obtained the following result, which gives a complete characterization of the discrete unipotent parabolic subgroups Γ of $\mathrm{PU}(n,1)$ for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold. In this statement, a nilpotent Lie subalgebra $\mathfrak{n}$ of the Lie algebra of G is naturally identified with a subspace of the tangent space of $\mathbb{H}^n_{\mathbb{C}}$ at a point o, and it is called totally real if $\mathfrak{n} \cap J\mathfrak{n} = 0$, where J is the complex structure on $T_o\mathbb{H}^n_{\mathbb{C}}$.

Theorem ([Mie24, Theorem 1.4])**.** Let Γ be a discrete unipotent parabolic subgroup of $\mathrm{PU}(n,1)$, N_Γ its Zariski closure, and $\mathfrak{n}_\Gamma$ the Lie algebra of N_Γ. Then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold if and only if $\mathfrak{n}_\Gamma$ is totally real.

We have extended this characterization to general parabolic subgroups (see Theorem 2.1). This provides a complete answer to our problem in the case of elementary, torsion-free subgroups of $\mathrm{PU}(n,1)$. Prior to Miebach's work, partial results had been obtained in the articles [dFab98; dFI01; Mie10; Che13]. Note that a discrete, virtually abelian, and torsion-free subgroup Γ of

$\mathrm{PU}(n,1)$ is necessarily elementary, and a consequence of our results is that if $\Gamma < \mathrm{PU}(n,1)$ is discrete, virtually abelian, and torsion-free, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold (see Corollary 2.8).

Among nonelementary subgroups, two nested families can be distinguished: *convex-cocompact* groups and *geometrically finite* groups, defined in Sections 1.2.5 and 1.2.6, respectively. For convex-cocompact subgroups, the question of holomorphic convexity of the quotients has been settled: Yue [Yue99], and later Dey and Kapovich [DK20], showed that the quotient of complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ by a convex-cocompact subgroup of $\mathrm{PU}(n,1)$ is always holomorphically convex. In this thesis, we resolve this question for geometrically finite subgroups. The quotient of $\mathbb{H}^n_{\mathbb{C}}$ by a geometrically finite subgroup Γ of $\mathrm{PU}(n,1)$ is not necessarily holomorphically convex. We obtain a complete characterization of geometrically finite quotients satisfying this property, which we now describe, referring to Proposition 2.10 for more complete results. The quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ has finitely many parabolic cusps, corresponding to conjugacy classes of maximal parabolic subgroups of Γ. We establish that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex if and only if for every maximal parabolic subgroup P of Γ, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is holomorphically convex (or, equivalently, Stein). The question of the absence of compact analytic subvarieties of positive dimension is very rich; there exist criteria valid for all discrete torsion-free subgroups of $\mathrm{PU}(n,1)$. These will be described later.

Before that, and to illustrate the depth of this last question, consider a faithful and discrete representation $\rho : \Gamma \to \mathrm{PU}(n,1)$, where Γ is the fundamental group of a closed surface of genus $g \geq 2$. One can associate to ρ a number $\tau(\rho)$ called the *Toledo invariant*, which can take only finitely many values between $2-2g$ and $2g-2$. We refer to Example 5 in Section 1.2.5 for more details and references. If ρ is a *maximal representation*, in the sense that $|\tau(\rho)| = 2g-2$, then the quotient $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ is holomorphically convex and contains a compact complex subvariety of dimension 1: it is therefore not a Stein manifold. If $\tau(\rho) = 0$, then $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ does not contain any compact analytic subvariety of positive dimension. Bronstein has constructed examples of convex-cocompact representations ρ with $\tau(\rho)$ strictly between 0 and $2g-2$, such that the quotient $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ contains a compact complex subvariety: hence it is holomorphically convex but not Stein [Bro25]. For these representations with intermediate Toledo invariant, it is conjecturally believed that the situation is mixed, with some quotients containing compact analytic subvarieties of positive dimension and others not — however, aside from Bronstein's examples, we are not aware of any case where it is known whether the quotient contains a compact analytic subvariety. In particular, the article [GKL01] contains examples for which this question remains open.

Beyond geometrically finite groups, we have very little information about the holomorphic convexity of the quotient (but see Proposition 2.2 for a result in this generality). However, we do have criteria that allow us to affirm that the quotient does not contain any compact analytic subvariety of positive dimension (see Proposition 2.7 for an exhaustive presentation of the known criteria). One such criterion, obtained by Dey and Kapovich [DK20], concerns the *critical exponent* of the group, which is a rather dynamical measure of the size of the orbits. More precisely, the critical exponent of a discrete subgroup Γ of $\mathrm{PU}(n,1)$ is the real number defined by

$$\delta(\Gamma) = \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty\}, \tag{1}$$

where o is any point in $\mathbb{H}^n_{\mathbb{C}}$ and d the complex hyperbolic distance on the ball, normalized so that the associated Riemannian metric has sectional curvature pinched between -4 and -1. This value does not depend on the choice of o. We refer to Section 1.2.7 for more details on this notion. Dey and Kapovich showed that if $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ does not contain any compact analytic subvariety of positive dimension. The article [DK20] also proposes the following three conjectures.

Conjecture 1 ([DK20, Conjecture 2])**.** Let Γ be a discrete torsion-free subgroup of $\mathrm{PU}(n,1)$. If $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Conjecture 2 ([DK20, Conjecture 17])**.** Let Γ be a convex-cocompact torsion-free subgroup of $\mathrm{PU}(n,1)$. If $\delta(\Gamma) = 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not Stein if and only if Γ acts cocompactly on a totally geodesic complex disk contained in $\mathbb{H}^n_{\mathbb{C}}$.

A group acting cocompactly on a totally geodesic complex disk contained in $\mathbb{H}^n_{\mathbb{C}}$ will be referred to as a *complex Fuchsian group* in what follows.

Conjecture 3 ([DK20, Conjecture 21])**.** Let Γ be a discrete torsion-free subgroup of $\mathrm{PU}(n,1)$. If $\delta(\Gamma) < 2k$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains no compact analytic subvariety of dimension greater than or equal to k.

As recalled above, Dey and Kapovich resolved the first conjecture in the setting of convex cocompact groups. Here is the current status of these conjectures: the first remains open, but we have resolved it in the case of parabolic subgroups (Theorem B) and geometrically finite subgroups (Theorem A) of $\mathrm{PU}(n,1)$. We have resolved the second one (Theorem C), and the third was resolved by Connell, McReynolds, and Wang in [CMW23, Theorem 1.5].

Among the open questions in the field, we may mention Conjecture 1 of Dey and Kapovich in full generality, but also a question originally raised by [Che13]: let Γ be a discrete torsion-free subgroup of $\mathrm{PO}(n,1)$ embedded in $\mathrm{PU}(n,1)$ so as to preserve a totally geodesic copy of the n-dimensional real hyperbolic space $\mathbb{H}^n_{\mathbb{R}}$. Is $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ a Stein manifold? It is easy to see that in this case $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains no compact analytic subvariety of positive dimension, and thus $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold if and only if it is holomorphically convex. Theorems A and B provide a positive answer to this question when Γ is elementary or geometrically finite. The following question is, to our knowledge, also open: given two elements $g, h \in \mathrm{PU}(n,1)$ such that the group Γ generated by g and h is discrete, can we state sufficient and *computable* conditions on g and h that imply that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold? The study of specific examples could also be interesting; for instance, the quotients constructed in [GKL01] and [Gra15], are convex-cocompact. Do they contain compact analytic subsets of positive dimension? Finally, one may wonder what happens when deforming the group Γ. If $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains no compact analytic subset of positive dimension, or if $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex, does a small deformation of Γ that remains discrete preserve this property?

We now state our main results concerning quotients of the complex ball.

Theorem A. *Let Γ be a geometrically finite and torsion-free subgroup of* $\mathrm{PU}(n,1)$.

(a) If Γ is Gromov-hyperbolic, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex.

(b) If Γ is a free group or if $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

(c) If Γ preserves a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Theorem B. *Let Γ be a discrete and torsion-free parabolic subgroup of* $\mathrm{PU}(n,1)$.

(a) If $\delta(\Gamma) < 2$ or if Γ preserves a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then Γ is virtually Abelian.

(b) If Γ is virtually Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Theorem C. *Let Γ be a discrete and torsion-free subgroup of* $\mathrm{PU}(n,1)$ *with $\delta(\Gamma) = 2$. Suppose that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety of positive dimension. Then Γ is a complex Fuchsian group.*

The proof of Theorem C relies crucially on *Patterson–Sullivan theory.* This theory was introduced for real hyperbolic space in the articles [Pat76] and [Sul79], and then developed in the more general context of CAT(-1) spaces in [Rob03]; see also [Cor90; CI99; Coo93]. For a complete simply connected Riemannian manifold X with curvature bounded above by -1, it consists in associating to a group Γ acting freely and properly discontinuously by isometries on X a family of finite measures $(\mu_x)_{x\in X}$ on the visual boundary of X, indexed by the points of X, and with good transformation properties under the action of Γ. This theory was first used to study properties of the geodesic flow on X/Γ, and then, for instance, to relate the critical exponent of Γ to the Hausdorff dimension of a certain subset of the boundary of X, called the conical limit set of X [Cor90; Pau97]. One of the advantages of approaching Dey and Kapovich's conjectures through Patterson–Sullivan theory is that the latter has been extended to higher rank symmetric spaces $\mathbb{X} = G/K$, first by Burger in rank 2 [Bur93], then by Albuquerque [Alb99] and by Quint [Qui02a; Qui02b]. See also [PS17] and [PSW23] for accounts of this theory.

More precisely, to establish Theorem C, our strategy consists in showing that, under appropriate assumptions, a certain function constructed in [CMW23] from Patterson–Sullivan theory is (strictly) plurisubharmonic, and then deducing geometric consequences from it. This proof uses the fact that for any complete simply connected Kähler manifold with sectional curvature less than or equal to -1, the Busemann functions B_ξ, with ξ in the visual boundary of X, satisfy the lower bound

$$i\partial\bar{\partial}B_\xi \geq \omega, \tag{2}$$

and this is in fact an equality for the complex hyperbolic space with curvature between -4 and -1. We refer to Section 1.2.4 for the definition and properties of these Busemann functions. Let us note that the function defined in [CMW23] from Patterson–Sullivan measures was also reused to study expansion properties of locally symmetric spaces in [FL24].

In higher rank, our approach consists in adapting this strategy by replacing the notion of critical exponent with the entropy associated to a certain linear form. We present in the following section the results obtained in this direction.

Quotients of higher rank Hermitian symmetric spaces (with C. Davalo)

We now turn to the study of quotients of Hermitian symmetric spaces $\mathbb{X} = G/K$ of noncompact type. In collaboration with Colin Davalo, we obtain results on the absence of compact analytic subvarieties of positive dimension in quotients of the form $\Gamma\backslash\mathbb{X}$[1], under a certain hypothesis on

[1] Since we write $\mathbb{X} = G/K$ for the symmetric space, we quotient on the left rather than on the right by Γ when working in higher rank.

the discrete subgroup Γ of G. This hypothesis takes the form of an inequality that generalizes the one proposed by Dey and Kapovich concerning the critical exponent. In short, the idea is to replace the critical exponent with the entropy associated to a linear form corresponding to the Shilov boundary of the symmetric space. We also obtain a characterization of the case of equality (analogous to Theorem C), which allows us to apply our results to a broad family of examples.

To fix notation, let G be a *simple Lie group of noncompact type*. In this thesis, this means that G is a non-Euclidean connected Lie group with finite center and simple Lie algebra. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a decomposition of its Lie algebra $\mathfrak{g}$ induced by a Cartan involution of $\mathfrak{g}$. We consider $\mathbb{X} = G/K$ the associated symmetric space, obtained by quotienting $\mathbb{X}$ by the maximal compact subgroup whose Lie algebra is $\mathfrak{k}$. We assume that G is *Hermitian*, meaning that there exists a complex (and even Kähler) manifold structure on $\mathbb{X}$ for which elements of G act by biholomorphic isometries on $\mathbb{X}$. Via the exponential map, the maximal abelian subalgebras $\mathfrak{a}$ of $\mathfrak{p}$ are in correspondence with the maximal flats of $\mathbb{X} = G/K$ passing through the point $eK \in \mathbb{X}$. We fix such a subalgebra $\mathfrak{a}$ along with a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. The Cartan decomposition $G = K \exp \mathfrak{a}^+ K$ of G then allows one to construct a Cartan projection $\mu \colon G \to \mathfrak{a}^+$. All these classical notions will be recalled in more details and with references in Section 1.3.

In the visual boundary $\partial_{\text{vis}}\mathbb{X}$ of $\mathbb{X}$, there exists a specific closed G-orbit, called the *Shilov boundary* of $\mathbb{X}$. It intersects the boundary of the cone $\exp(\mathfrak{a}^+)$ in exactly one point ξ, and there exists a unique simple root $\breve{a} \in \mathfrak{a}^*$ which does not vanish on the open half-line from the origin to this point. This root will be called the root associated with the Shilov boundary of $\mathbb{X}$. We refer to Definition 1.44 for more details.

The hypothesis generalizing that of Dey and Kapovich in higher rank involves the entropy of a discrete subgroup associated with this simple root $\breve{a}$. The entropy associated to a linear form $\phi \in \mathfrak{a}^*$ was introduced in [Bur93; Alb99; Qui02a; Qui02b]. Unlike in the previous section, we prefer here to work with a faithful and discrete representation $\rho \colon \Gamma \to G$ of a finitely generated group Γ into G, rather than with a finitely generated discrete subgroup of G. The entropy of ρ associated with the root $\breve{a}$ is the number defined by

$$h_\rho(\breve{a}) := \inf \left\{ s \in \mathbb{R} \;\middle|\; \sum_{\gamma \in \Gamma} e^{-s\breve{a}(\mu(\rho(\gamma)))} < +\infty \right\} \in \mathbb{R} \cup \{+\infty\}.$$

Also notice that Formula (1) also defines a notion of critical exponent $\delta(\Gamma)$ in this setting. We are now able to state our main result in higher rank, obtained in collaboration with Colin Davalo.

Theorem D. *Let $\rho \colon \Gamma \to G$ be a discrete and faithful representation of a torsion-free group Γ into a classical simple Hermitian Lie group G of noncompact type. Assume that the critical exponent $\delta(\rho(\Gamma))$ of $\rho(\Gamma)$ is nonzero. Fixing a maximal flat $\exp(\mathfrak{a})$ inside the symmetric space $\mathbb{X}$ associated with G, with $\mathfrak{a} \subset \mathfrak{p}$, let $\breve{a} \in \mathfrak{a}^*$ be the root associated with the Shilov boundary of $\mathbb{X}$ and let $h_\rho(\breve{a})$ denote the entropy of ρ associated with the root $\breve{a}$.*

1. *If $h_\rho(\breve{a}) < 1$, then $\rho(\Gamma)\backslash\mathbb{X}$ does not contain any compact analytic subvariety of positive dimension.*
2. *If $h_\rho(\breve{a}) = 1$ and $\rho(\Gamma)$ is nonelementary, then $\rho(\Gamma)\backslash\mathbb{X}$ contains a compact analytic subvariety of positive dimension if and only if Γ is the fundamental group of a surface and $\rho(\Gamma)$ preserves a totally geodesic complex disk in $\mathbb{X}$.*

The meaning of "nonelementary" will be given in Section 1.3.5 and the root $\breve{\alpha}$ associated with the Shilov boundary of $\mathbb{X}$ is the content of Definition 1.44. In fact, the conclusion in Theorem D-2 can be strengthened: the totally geodesic complex disk preserved by $\rho(\Gamma)$ is a *diagonal disk*, that is the image of the diagonal by the totally geodesic embedding of a maximal polydisk (see Definition 2.13).

It was shown in [PSW23, Theorem 9.8] that $h_\rho(\breve{\alpha}) = 1$ for any maximal representation ρ of the fundamental group $\pi_1(S_g)$ of a surface of genus $g \geq 2$ into G. From Theorem D, we deduce the following corollary.

Corollary E. *Let G be a simple Hermitian Lie group of noncompact type and $\mathbb{X}$ its associated symmetric space. Let $\rho\colon \pi_1(S_g) \to G$ be a maximal representation. Then $\rho(\Gamma)\backslash\mathbb{X}$ contains a compact analytic subvariety of positive dimension if and only if $\rho(\Gamma)$ preserves a diagonal disk of $\mathbb{X}$.*

As previously explained, the strategy used to prove Theorem D is an elaboration of the arguments used to prove Theorem C. In the setting of Cartan–Hadamard manifolds, Busemann functions are convex (see [EO73]), and consequently, for a Hermitian symmetric space of non-compact type, Busemann functions are plurisubharmonic (we refer to Section 1.1.1 for the link between convexity and plurisubharmonicity in Kähler geometry). In general, they are not strictly plurisubharmonic and do not satisfy the inequality (2). A step of independent interest in the proof of Theorem D is a lower bound on the Levi form of Busemann functions in higher rank, which improves upon [Eys97, Proposition 4.4.4] and is based on the computation of the Hessian of Busemann functions for symmetric spaces carried out in [Dav23].

Proposition F. *Let G be a simple Hermitian Lie group of noncompact type and $(\mathbb{X}, \omega)$ its associated symmetric space viewed as a Kähler manifold. Fixing a maximal flat $\exp(\mathfrak{a})$ of $\mathbb{X}$ with $\mathfrak{a} \subset \mathfrak{g}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\breve{\alpha} \in \mathfrak{a}^*$ be the root associated with the Shilov boundary of $\mathbb{X}$.*

Let $w \in \mathfrak{a}^+$ be another unit vector, and let ξ_w be the point on the visual boundary of $\mathbb{X}$ associated with w. Choose a Busemann function $B_{\xi_w}\colon \mathbb{X} \to \mathbb{R}$ at the point ξ_w. Two such functions differ only by an additive constant, so their Levi form is independent of this choice. It satisfies the inequality:

$$i\partial\bar{\partial}B_{\xi_w} \geq \frac{\breve{\alpha}(w)}{2}\omega.$$

Moreover, equality holds if and only if ξ_w belongs to the Shilov boundary of $\mathbb{X}$.

In particular, the Busemann function B_{ξ_w} is strictly plurisubharmonic whenever $\breve{\alpha}(w) \neq 0$, that is, outside a face of the simplex obtained by projectivizing the Weyl chamber, which is the face opposite to the vertex corresponding to the Shilov boundary of $\mathbb{X}$. For a point ξ_w belonging to this face, the Busemann function B_{ξ_w} is not strictly plurisubharmonic, as will be seen in the proof of Proposition F.

The Kähler form ω induced by the Hermitian structure on $\mathbb{X}$ is only defined up to a positive scalar multiple. One can verify that the inequality obtained in the proposition is homogeneous under rescaling of the Kähler form. In coherence with Inequality (2), one can verify that the metric satisfying $\breve{\alpha}(\breve{w}) = 2$ is the unique metric on $\mathbb{X}$ such that:

$$\sup_{v \in T\mathbb{X}} \mathrm{HSC}(v) = -4,$$

where HSC denotes the *holomorphic sectional curvature* of $\mathbb{X}$, see [CØ03].

Toroidal compactifications of ball quotients

We now describe another aspect of our work, whose guiding theme is the following:

Given a complex manifold X that is not Stein (typically a compact manifold), show under suitable hypotheses that the universal cover of X is Stein.

In dimension 1, the uniformization theorem implies that the universal cover of a compact Riemann surface is either compact or a Stein manifold; in both cases it is holomorphically convex. Shafarevich's conjecture, formulated in 1972 for projective varieties, predicts that this fact remains true in higher dimension:

Conjecture 4 (Shafarevich, [Sha13])**.** The universal cover of any compact Kähler manifold is holomorphically convex.

This conjecture is open in general, and it is not even known whether a non-constant holomorphic function always exists on the universal cover of a compact Kähler manifold with infinite fundamental group. It is, however, known when the fundamental group of the variety is linear [Eys+12; CCE15], as well as for certain specific examples [EM15]. See also [Kat97; KR98; Eys04] for earlier results, and [Eys11] for a survey on the conjecture. Finally, as already noted, the statement is known for a compact Kähler manifold with nonpositive sectional curvature, since its universal cover is Stein.

There is a family of Kähler manifolds called *toroidal compactifications of ball quotients*, or sometimes *Mumford compactifications*, first described in [Mum75; Mok12; Ash+10]. They are obtained by compactifying the quotient of the ball $\mathbb{H}^n_{\mathbb{C}}$ by a non-uniform and torsion-free lattice Γ whose maximal parabolic subgroups are unipotent. For every non-uniform lattice Γ_0 in $\mathrm{PU}(n,1)$, there exists a finite-index subgroup $\Gamma < \Gamma_0$ satisfying this assumption, see [Hum98]. We denote by X_Γ the toroidal compactification of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. It is obtained by "gluing" a complex torus of dimension $n-1$ at the end of each cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, and its construction is detailed in Section 3.1. Toroidal compactifications are in fact projective varieties [Mok12; DD15], and several authors have studied their properties from the viewpoint of algebraic geometry [DD17; BT18; Cad21; Mem23], as well as from other perspectives [Py17; DS18]. Whether the fundamental group of a toroidal compactification is linear remains unknown.

There are three types of results concerning these varieties. Some hold for every X_Γ. Others become true only starting from a certain dimension (for instance, if $n \geq 6$, then the canonical bundle of X_Γ is ample, which is known to be false for some toroidal compactifications in dimension 2, see [BT18]). Finally, certain results show that for every non-uniform lattice Γ_0 in $\mathrm{PU}(n,1)$, there exists a finite-index subgroup Γ' whose parabolic subgroups are unipotent and such that, for every finite-index subgroup $\Gamma'' < \Gamma'$, a certain property P holds for the toroidal compactification $X_{\Gamma''}$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$. For example, the following result of Hummel and Schroeder, concerning the metric study of these varieties, is of great importance for our work.

Theorem ([HS96, Theorems 2 and 7])**.** Let Γ be a torsion-free non-uniform lattice in $\mathrm{PU}(n,1)$ whose parabolic subgroups are unipotent. Then X_Γ is a Kähler manifold. Moreover, there exists a finite index subgroup $\Gamma' < \Gamma$ such that, for every finite index subgroup $\Gamma'' < \Gamma'$, the toroidal compactification $X_{\Gamma''}$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ admits a Riemannian metric with nonpositive sectional curvature.

Whether toroidal compactifications satisfy the Shafarevich conjecture remains an open question. A more accessible question is likely the following. Let Γ_0 be a non-uniform and torsion-free lattice in $\mathrm{PU}(n,1)$. Does there exist a finite index subgroup Γ' of Γ_0 whose parabolic subgroups are unipotent and such that for every finite index subgroup Γ'' of Γ', the toroidal compactification $X_{\Gamma''}$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ satisfies the Shafarevich conjecture? We obtain a positive answer to this second question under the additional assumption that Γ_0 is an arithmetic lattice. Notice that, after possibly replacing Γ' by a finite index subgroup, this question is equivalent to asking whether the universal cover of $X_{\Gamma''}$ is Stein. Indeed, this follows from the already cited result of Hummel and Schroeder, which implies that one can find a finite index subgroup $\Gamma' \subset \Gamma_0$ such that for every finite index subgroup $\Gamma'' \subset \Gamma'$, the universal cover of $X_{\Gamma''}$ is contractible and, in particular, does not contain any compact analytic subvariety of positive dimension.

Before this, and independently, we obtain results that refine and clarify the theorem of Hummel and Schroeder. We show that the toroidal compactifications X_Γ do not admit any Kähler metric with nonpositive sectional curvature (Theorem G), thus confirming a claim from [HS96], but that they do admit a Kähler metric with nonpositive holomorphic bisectional curvature (Theorem H). This allows us to positively answer a question raised by Diverio in [Div22] on the existence of a compact complex manifold admitting no Kähler metric with strictly negative holomorphic sectional curvature but admitting a Kähler metric with (bi)sectional holomorphic curvature that is *quasi-negative* in the sense of [DT19, Definition 1.1], meaning that the holomorphic bisectional curvature is everywhere nonpositive and negative at some point (Corollary I).

We then show that the universal cover $\widetilde{X_\Gamma}$ of X_Γ is a Stein manifold when the lattice Γ is arithmetic and "sufficiently deep". To do this, we study the properties of the Albanese map of varieties obtained by compactifying finite covers of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, and we obtain results concerning the restriction of this map to the tori added during the compactification (Theorem J), extending earlier work by Eyssidieux [Eys18]. This refinement allows us to show that, for every non-uniform and torsion-free arithmetic lattice Γ_0 there exists a finite index subgroup $\Gamma' < \Gamma_0$ whose parabolic subgroups are unipotent and such that for every finite index subgroup $\Gamma'' < \Gamma'$, the universal cover of the toroidal compactification of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ is a Stein manifold (Theorem K). When $n = 2$, this result was previously obtained by Eyssidieux [Eys18].

Let us now describe these results more precisely. In [HS96], Hummel and Schroeder assert that toroidal compactifications do not admit any Kähler metric with nonpositive sectional curvature. A proof of this statement in complex dimension 2 for "sufficiently deep" lattices was given by Di Cerbo in [Di 12, Theorem A]. Generalizing this approach to arbitrary dimension, we prove the following result.

Theorem G. *Let Γ be a non-uniform and torsion-free lattice in $\mathrm{PU}(n,1)$ for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification X_Γ. Then there is no Kähler metric with nonpositive sectional curvature on X_Γ.*

In particular, the techniques for constructing Kähler metrics on X_Γ developed in [HS96] do not yield a metric with nonpositive sectional curvature. However, we show through computation that they are sufficient to construct a Kähler metric whose curvature satisfies a weaker non-positivity condition, namely, nonpositive holomorphic bisectional curvature.

Theorem H. *Let Γ_0 be a non-uniform and torsion-free lattice in $\mathrm{PU}(n,1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a Kähler metric with nonpositive holomorphic bisectional curvature.*

The metric constructed in the proof of this theorem has negative holomorphic bisectional curvature on the open subset $\mathbb{H}^n_{\mathbb{C}}/\Gamma'' \subset X_{\Gamma''}$. In particular, it is quasi-negative. In their article, Diverio and Trapani showed that every compact Kähler manifold with quasi-negative holomorphic sectional curvature has an ample canonical bundle, a result already known under the assumption of negative holomorphic sectional curvature [WY16, Theorem 2], [TY17, Corollary 1.3]. This answered a question posed by Yau. Naturally, this led Diverio to ask in [Div22] whether there exist compact Kähler manifolds with quasi-negative holomorphic sectional curvature that do not admit any Kähler metric with negative holomorphic sectional curvature. Since toroidal compactifications contain tori, which are not Kobayashi hyperbolic, they provide a positive answer to this question.

Corollary I. *For Γ'' a non-uniform lattice as in Theorem H, the complex manifold X''_Γ is an example of a closed manifold which admits a Kähler metric of quasi-negative holomorphic bisectional curvature, but which does not admit a Kähler metric of negative holomorphic sectional curvature.*

We further notice that Theorem H, combined with the main result of [DT19], provides an alternative proof of [DD17, Theorem 1.3]: for a non-uniform lattice Γ'' as in Theorem H, the canonical bundle of X''_Γ is ample. For a more direct proof, we note in Section 3.3 that the metric constructed in the proof of Theorem H has negative Ricci curvature. Moreover, applying [Gue22, Theorem 3.1], we obtain that every irreducible subvariety of X''_Γ not contained in the complement of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ is of general type.

We now state our results related to the Shafarevich conjecture for the toroidal compactifications X_Γ. Let Γ_0 be a non-uniform and torsion-free arithmetic lattice in $\mathrm{PU}(n,1)$. Eyssidieux has shown that there exists a finite index subgroup $\Gamma' < \Gamma_0$ whose parabolic subgroups are unipotent and such that for every finite index subgroup $\Gamma'' < \Gamma'$, the Albanese map $A_{\Gamma''}\colon X_{\Gamma''} \longrightarrow \mathrm{Alb}(X_{\Gamma''})$ of the toroidal compactification $X_{\Gamma''}$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma''$ denoted by $A_\Gamma\colon X_\Gamma \longrightarrow \mathrm{Alb}(X_\Gamma)$, is an immersion on the complement of the tori added during the compactification [Eys18, Theorem 1.3]. He then concluded in dimension 2 that the universal cover of X_Γ is holomorphically convex, using a result of [Nap90] valid only in this dimension.

To study the higher dimensional case, we make Eyssidieux's result about Albanese mappings more precise. To do so, we analyse the behaviour of the Albanese map on the tori added during the compactification, which we will call in the following "boundary tori of X_Γ". More precisely, a boundary torus of X_Γ is a connected component of the complement of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ in X_Γ. With this terminology, we prove:

Theorem J. *Let Γ_0 be a non-uniform and torsion-free arithmetic lattice in* $\mathrm{PU}(n,1)$. *There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ satisfies the following properties:*

- *Its Albanese map A_Γ is an immersion on the open subset $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$.*
- *For any boundary torus T of X_Γ, the map $A_{\Gamma|T}$ is either an immersion or a constant.*

The arithmeticity of Γ_0 plays a crucial role in the proof of Theorem J, as in [Eys18]. Determining whether, for every boundary torus T of X_{Γ_0}, there exists a finite index subgroup $\Gamma < \Gamma_0$ such that the Albanese map A_Γ of X_Γ is an immersion when restricted to the tori above T seems much more difficult. However, the weaker result stated in Theorem J is sufficient for our purposes. We combine it with the arguments from [Nap90], which apply in all dimensions, concerning holomorphic convexity with respect to a sufficiently positive line bundle, and we adapt the reasoning from that article in dimension 2 to generalize Eyssidieux's result to all dimensions:

Theorem K. *Let Γ_0 be a non-uniform and torsion-free arithmetic lattice in* $\mathrm{PU}(n,1)$. *Then there exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the universal cover $\widetilde{X_\Gamma}$ of the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.*

This theorem follows from Proposition 3.15 stated in Section 3.5, which is a more general result concerning projective varieties that are the domain of a generically finite holomorphic map to a compact variety whose universal cover is Stein.

Summary of chapters

This thesis consists of three chapters.

- The first chapter introduces the preliminary notions required to prove the results stated in the first two sections of this introduction. In the first section, we begin by recalling the notions from complex analysis that are used (plurisubharmonicity, analytic subvarieties, holomorphic convexity, Stein manifolds). Then, in Section 1.2, we describe in detail the complex hyperbolic space and its biholomorphism group $\mathrm{PU}(n,1)$. We introduce the notions of limit set and domain of discontinuity, and we discuss the structure of the stabilizer of a point at infinity in complex hyperbolic space. We define convex cocompact subgroups and geometrically finite subgroups of $\mathrm{PU}(n,1)$. We introduce the notion of critical exponent, briefly describe the Patterson–Sullivan theory in rank 1. Finally, in Section 1.3, we present some structural results about higher rank Hermitian symmetric spaces that will be used later: these include the root decomposition of a simple Lie group of noncompact type, the notions of Weyl chambers and Cartan projection, the visual boundary of the associated symmetric space, higher rank Busemann functions, and higher rank Patterson–Sullivan theory, developed notably by Quint in [Qui02a; Qui02b] via the notions of growth indicator function and entropy.

- The second chapter, in Section 2.1, contains the proofs of results concerning ball quotients. We begin by stating results that hold in the more general setting of simply connected complete Kähler manifolds with sectional curvature less than or equal to -1, and we present criteria ensuring that the quotient of such a manifold does not contain any compact analytic subvariety of positive dimension (Proposition 2.7). Then, we provide a complete characterization of discrete parabolic subgroups Γ of $\mathrm{PU}(n,1)$ for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold (Theorem 2.1), we prove Theorem B and give examples of parabolic ball quotients. Moving on to geometrically finite quotients, we state and prove the characterization mentioned earlier of geometrically finite subgroups Γ for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex (Proposition 2.10), and then prove Theorem A. In Section 2.2, we present the proofs of results obtained in collaboration with C. Davalo in the higher rank setting (Theorems D and F).

- The third chapter concerns toroidal compactifications of ball quotients. In Section 3.1, we recall their construction, beginning with a description of the quotient of the ball by a nontrivial element in the center of the unipotent radical of $\mathrm{PU}(n,1)$. In Section 3.2, we prove a general result (Proposition 3.4) that implies Theorem G. Then, in Section 3.3, we construct a Kähler metric with quasi-negative holomorphic bisectional curvature and negative Ricci curvature on toroidal compactifications of ball quotients by "sufficiently deep" lattices (Theorem H). We then turn to our results concerning the Shafarevich conjecture for toroidal compactifications in Sections 3.4 and 3.5, which respectively contain the proofs of Theorems J and K.

Chapter 1

Preliminary concepts

We introduce here some notions that will be needed in the following chapters. In the first section, which is about complex analysis, we define the notions of plurisubharmonicity, analytic subvarieties, holomorphic convexity and Stein manifolds, which are central to this thesis. We illustrate these notions with Proposition 1.12, which implies that suitable convex-cocompact quotients are holomorphically convex. The second section is an introduction to the complex hyperbolic space and the discrete subgroups of $\mathrm{PU}(n,1)$. In particular, we define Busemann functions, and we introduce the theory of Patterson–Sullivan in this framework. The third section is a very brief account of the theory of Hermitian symmetric spaces of higher rank, that also contains a description of the higher rank Patterson–Sullivan theory from the viewpoint of Quint's growth indicator function, developped in [Qui02a] and [Qui02b].

1.1 Notions from complex analysis

Our main references for this section are [Dem; Hör90; Ber10].

1.1.1 Convexity, Levi form, and plurisubharmonicity

One of the fundamental analytic concepts in this work is the notion of a plurisubharmonic function on a complex manifold. A real manifold X does not admit any canonical second-order differential operator. To define such an operator, one must endow the manifold with an additional structure. For instance, a Riemannian metric allows to define the *Hessian* of a smooth function $f\colon X \to \mathbb{R}$ using the Levi-Civita connection ∇ via the identity

$$D^2(f)(X,Y) = X(Yf) - (\nabla_X Y)f.$$

Endowing X with a complex structure J allows one to decompose the exterior derivative d (or more precisely, its complexification $d \otimes_{\mathbb{C}} id\colon C^\infty(X,\mathbb{C}) \to \Omega^1(X,\mathbb{C})$) as a sum of two operators ∂ and $\bar\partial$, whose composition $\partial\bar\partial$ is non zero. If $f\colon X \to \mathbb{R}$ is a smooth real-valued function on X, then the alternating form $i\partial\bar\partial f$ is real-valued, and one obtains a quadratic form called the *Levi form of f* and denoted $\mathrm{Lev}(f)$ by setting

$$\mathrm{Lev}(f)(v,v) := 2i\partial\bar\partial f(v, Jv).$$

In local holomorphic coordinates $(z^1, \dots, z^n)$, the Levi form of f takes the form

$$\mathrm{Lev}(f)(v, v) = 4 \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial z_i \partial \overline{z_j}} v_i \overline{v_j}.$$

Some authors define the Levi form using the constant $1/2$ instead of 2, which eliminates the factor 4 in the above expression; others use the constant 1, which introduces a factor 2 on the left-hand side of equation (1.1) below. To avoid ambiguity, we will systematically write $2i\partial\bar{\partial}f(v, Jv)$ instead of $\mathrm{Lev}(f)(v, v)$ starting from the next chapter. A function $f \colon X \to \mathbb{R}$ of class C^2 is said to be *plurisubharmonic* (resp. *strictly plurisubharmonic*), if its Levi form is positive (resp. positive definite). This notion, which requires no metric, extends to continuous functions. A continuous function $f \colon X \to \mathbb{R}$ is called plurisubharmonic (resp. strictly plurisubharmonic), if for every $x \in X$, there exists a chart $\phi \colon U \subset X \to V \subset \mathbb{C}^n$ containing x and a constant $\epsilon = 0$ (resp. $\epsilon > 0$), such that the function $g := f \circ \phi^{-1} - \epsilon \|\cdot\|^2 \colon V \to \mathbb{R}$ satisfies the following property:

$$\forall a \in V, \forall \xi \in \mathbb{C}^n \text{ such that } |\xi| < d(a, {}^cV), \quad g(a) \leq \frac{1}{2\pi} \int_0^{2\pi} g(a + e^{i\theta}\xi) d\theta,$$

see [Dem, Section I.5]. If f is of class C^2, these definitions coincide with the ones given above in terms of the Levi form of f.

The notion of plurisubharmonicity is sometimes thought of as a complex analog of convexity. We now explain why, and begin by recalling several notions of convexity in Riemannian geometry. First, a continuous function f defined on an interval $I \subset \mathbb{R}$ is said to be *strictly convex* if for all distinct points $x, y \in I$ and all $t \in]0, 1[$, we have $f((1-t)x + ty) < (1-t)f(x) + tf(y)$. It is said to be *strongly convex* if there exists a constant $m > 0$ such that the function $t \mapsto f(t) - mt^2$ is convex. On a Riemannian manifold (M, g), a continuous function $f \colon M \to \mathbb{R}$ is said to be convex, respectively strictly convex, respectively strongly convex, if for every geodesic segment $\gamma \colon [0, 1] \to M$, the function $f \circ \gamma \colon [0, 1] \to \mathbb{R}$ has the same property. Moreover, if f is C^2, then f is convex if and only if its Hessian $D^2(f)$ is positive semi-definite, and f is strongly convex if and only if $D^2(f)$ is positive definite.

In the analogy between these two notions, plurisubharmonic functions correspond to convex functions, and *strictly* plurisubharmonic functions correspond to *strongly* convex functions (not to strictly convex ones). This analogy becomes explicit in the setting of Kähler manifolds, where the complex structure and the Riemannian metric are compatible. Let (X, ω) be a Kähler manifold and $f \colon X \to \mathbb{R}$ be a C^2 function. Then the Levi form of f is related to its Hessian by

$$\forall v \in TX, \quad \mathrm{Lev}(f)(v, v) = D^2(f)(v, v) + D^2(f)(Jv, Jv), \tag{1.1}$$

see [GW73, Lemma p.646] or [CE12, Formula (2.8)]. In particular, if f is convex, respectively strongly convex, then f is plurisubharmonic, respectively strictly plurisubharmonic. Greene and Wu showed that this remains true even without regularity assumptions on the function f.

Theorem 1.1 ([GW73]). *Let (X, ω) be a Kähler manifold and $f \colon X \to \mathbb{R}$ be a continuous function. If f is convex, then f is plurisubharmonic.*

We deduce the following useful corollary.

Corollary 1.2. *Let (X, ω) be a Kähler manifold and $f \colon X \to \mathbb{R}$ be a continuous function. If f is strongly convex, then f is strictly plurisubharmonic.*

Proof. Let x be a point of X and $(z_1, \dots, z_n)$ be holomorphic coordinates defined on a neighborhood Ω of x. Then for any open subset V such that $\overline{V} \subset \Omega$, there exists a constant $c > 0$ such that the function $z \mapsto f(z) - c\|z\|^2$ is convex, and hence plurisubharmonic on V. Thus f is strictly plurisubharmonic. □

The proof of Greene and Wu's theorem relies on approximation results. In the sequel we will use the following approximation theorem, which requires strict plurisubharmonicity. For a more general version of this result, we refer to [Dem].

Theorem 1.3 (Richberg, [Dem, Theorem I.5.21]). *Let X be a complex manifold, $f\colon X \to \mathbb{R}$ be a continuous plurisubharmonic function, and Ω be an open subset of X on which f is strictly plurisubharmonic. Then for every positive real number ϵ, there exists a plurisubharmonic function $g \in C^0(X) \cap C^\infty(\Omega)$, which is strictly plurisubharmonic on Ω and such that $|g - f| < \epsilon$ on X.*

1.1.2 Analytic subvarieties of a complex manifold

In this section, we give the definition of analytic subvarieties, we present the maximum principle for plurisubharmonic functions on these spaces, and we show that a Kähler manifold with exact Kähler form does not contain any compact analytic subvariety of positive dimension. Our main reference is [Chi89].

Definition 1.4. Let X be a complex manifold. A closed subset A of X is an *analytic subvariety* of X if for every point $x \in A$ there is a neighborhood U_x of x in X and a family of holomorphic functions $f_1, \dots, f_N\colon U_x \to \mathbb{C}$ such that

$$A \cap U = \{x \in U \mid f_1(x) = \cdots = f_N(x) = 0\}.$$

If moreover A is not discrete in X, we say that A is an analytic subvariety of *positive dimension.*

Proposition 1.5 (Maximum principle, [Chi89, Section 1.6]). *Let A be a connected analytic subvariety of a complex manifold X and $f\colon X \to \mathbb{R}$ be a plurisubharmonic function. If $f|_A$ achieves its maximum at some point of A, then $f|_A$ is constant.*

From this proposition, we deduce the following useful corollary.

Corollary 1.6. *Let X be a complex manifold. If there exists a strictly plurisubharmonic continuous function $f\colon X \to \mathbb{R}$, then X does not contain any compact analytic subvariety of positive dimension.*

A Kähler manifold with exact Kähler form cannot contain any compact analytic subvariety, as the following lemma shows.

Lemma 1.7. *Let (X, ω) be a Kähler manifold. If ω is exact on X, then X does not contain any compact analytic subvariety of positive dimension.*

Proof. We will use here the notion of dimension for analytic subvarieties, for which we refer to [Chi89, Section 1.2]. The existence of a compact analytic subvariety A of dimension $k \geq 1$ would force the cohomology class $[\omega]$ of ω to be nonzero, according to the formula:

$$\mathrm{Vol}(A) = \frac{1}{k!}[\omega]^k \cdot [A] > 0,$$

in which $[A]$ denotes the fundamental class of A viewed in $H_{2k}(X, \mathbb{R})$, see [Chi89, §14]. □

1.1.3 Holomorphic convexity

1.1.3.1 Definition

We begin by defining what a holomorphically convex manifold is. Let X be a complex manifold, and K be a compact subset of X. The *envelope of holomorphy* $\widehat{K}$ of K is the closed subset of X defined by

$$\widehat{K} := \{x \in X \mid \forall f\colon X \to \mathbb{C} \text{ holomorphic}, \ |f(x)| \leq \max_K |f|\}.$$

For example, the envelope of holomorphy of the unit sphere in $\mathbb{C}^2$ is the closed unit ball, and its envelope in $\mathbb{C}^2 \setminus \{0\}$ is the punctured unit ball, by Hartogs's extension theorem, see [Kra01, Theorem 1.2.6]. A complex manifold X is said to be *holomorphically convex* if the envelope of holomorphy of every compact subset of X is compact. This is equivalent to Property (**HC**) given in the introduction: for every sequence $(x_n)_{n\in\mathbb{N}}$ of points in X without accumulation point, there exists a holomorphic function $f\colon X \to \mathbb{C}$ such that

$$\sup_n |f(x_n)| = +\infty.$$

To show that a manifold is holomorphically convex, we will use throughout this work results due to Grauert on relatively compact open subsets with strictly pseudoconvex boundary inside a complex manifold. Recall that an open subset M of a (real) manifold X is said to have *smooth boundary* if there exists a smooth function $\phi\colon X \to \mathbb{R}$ such that $M = \phi^{-1}(]-\infty, 0[)$, $\partial M = \phi^{-1}(0)$, and such that $d\phi(x) \neq 0$ for every $x \in \partial M$. We then say that ϕ globally defines ∂M. Being a domain with smooth boundary is a local property: M has smooth boundary if and only if for every $x \in \partial M$, there exists an open neighborhood U of x in X and a function $\phi\colon U \to \mathbb{R}$ such that $M \cap U = \{\phi < 0\}$, $\partial M \cap U = \{\phi = 0\}$, and $d\phi(x) \neq 0$. We then say that ϕ defines ∂M at x. If M is a domain with smooth boundary in a complex manifold X, then the real hypersurface ∂M is not a complex submanifold, but one can still define a complex subbundle $T^{\mathbb{C}}\partial M$ of $T\partial M$ by setting $T^{\mathbb{C}}\partial M := T(\partial M) \cap J(T(\partial M))$. A relatively compact open subset $M \subset X$ is said to have *strictly pseudoconvex boundary* if it satisfies one of the equivalent statements in the following lemma, whose proof is classical.

Lemma 1.8. *Let X be a complex manifold and $M \subset X$ be a relatively compact open set. The following statements are equivalent:*

1. *For every $x \in \partial M$, there exists an open neighborhood U of x and a function $\phi\colon U \to \mathbb{R}$ defining ∂M at x which is strictly plurisubharmonic on $T^{\mathbb{C}}_x\partial M$, in the sense that:*

$$\forall v \in T^{\mathbb{C}}_x\partial M \setminus \{0\}, \ Lev(\phi)_x(v, v) > 0.$$

2. *M has smooth boundary, and every function $\phi\colon U \to \mathbb{R}$ defined on an open neighborhood U of a point $x \in \partial M$ and defining ∂M at x, is strictly plurisubharmonic on $T^{\mathbb{C}}_x\partial M$ in the sense of the previous point.*

3. *There exists a function $\phi\colon X \to \mathbb{R}$ globally defining ∂M that is strictly plurisubharmonic on ∂M, in the sense that:*

$$\forall x \in \partial M, \ \forall v \in T_xM \setminus \{0\}, \ Lev(\phi)_x(v, v) > 0.$$

Grauert's theorem can be stated as follows.

Theorem 1.9 ([Gra58, Theorem 1]). *Let X be a complex manifold and $M \subset X$ a relatively compact open subset of X with strictly pseudoconvex boundary. Then M is holomorphically convex.*

We will in fact use the following version of Grauert's theorem, which follows from the original one together with Runge-type approximation results. Before stating this alternative version, we recall the definition of an *exhaustion function*.

Definition 1.10. On a topological space X, a continuous function $f\colon X \to \mathbb{R}$ is called an *exhaustion* if all its sublevel sets $\{x \in X \mid f(x) < c\}$ with $c \in \mathbb{R}$ are relatively compact.

Theorem 1.11 ([Pet94]). *Let X be a complex manifold. Suppose that there exists a smooth exhaustion function $\phi\colon X \to \mathbb{R}$ which is plurisubharmonic on X and strictly plurisubharmonic outside a compact subset of X. Then X is holomorphically convex.*

Here are some remarks about this last theorem. First, a manifold satisfying its hypotheses is called a 1-convex manifold ([Dem, Definition IX.2.7]). Moreover, the assumption that ϕ is smooth is superfluous due to Theorem 1.3. The plurisubharmonicity assumption of ϕ is also superfluous, since if ϕ is an exhaustion function that is strictly plurisubharmonic outside a compact K, then K is contained in a sublevel $\{\phi < c\}$, and one can compose ϕ with a convex increasing function $\lambda\colon \mathbb{R} \to \mathbb{R}$ that is identically zero on $]-\infty, c[$ and strictly increasing on $]c, +\infty[$ to obtain a plurisubharmonic exhaustion which is strictly plurisubharmonic outside K. Furthermore, the conclusion can be strengthened: X is actually a *modification of a Stein space*. In particular not all holomorphically convex manifolds are 1-convex, as shown by the example of $\mathbb{CP}^1 \times \mathbb{C}$ or more generally the product of any compact manifold with an affine space $\mathbb{C}^n$.

1.1.3.2 Holomorphic convexity and compact convex subsets in negative curvature

Here is an example of an application of Grauert's results, taken from [Sar24], which improves and generalizes [DK20].

Proposition 1.12. *Let (X, ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and let Γ be a group acting freely and properly discontinuously by holomorphic isometries on X. Suppose that there exists a closed geodesically convex subset C of X that is Γ-invariant. Then the compact connected analytic subvarieties of X/Γ of positive dimension are contained in C/Γ. If furthermore the action of Γ on C is cocompact, then X/Γ is holomorphically convex.*

To prove this result, we begin with the following lemma.

Lemma 1.13. *Let (X, ω) be a Kähler manifold. Suppose that there exists a continuous function $\phi : X \to \mathbb{R}$ that is convex on X and strongly convex on $X \setminus \phi^{-1}(0)$, for the Riemannian metric $g = \Re(\omega)$. Then the compact connected subvarieties of X of positive dimension are contained in the level set $\phi^{-1}(0)$. If furthermore ϕ is an exhaustion function, then X is holomorphically convex.*

Proof. Let $\phi\colon X \to \mathbb{R}$ be a continuous function that is convex on X, and strongly convex on $X \setminus \phi^{-1}(0)$. By Theorem 1.1 and its Corollary 1.2, we get that ϕ is plurisubharmonic on X and strictly plurisubharmonic on $X \setminus \phi^{-1}(0)$. Moreover, for any compact connected subvariety A of X, the function $\phi|_A$ is constant by the maximum principle (see Proposition 1.5), so A is contained either in $\phi^{-1}(0)$ or in $X \setminus \phi^{-1}(0)$. Note that A cannot be contained in $X \setminus \phi^{-1}(0)$ because in that case, $\phi|_A$ would be constant and strictly plurisubharmonic. Thus, $A \subset \phi^{-1}(0)$. Assume now that

ϕ is an exhaustion. Then by Richberg's Theorem 1.3, there exists a continuous plurisubharmonic function $\widetilde{\phi}\colon X \to \mathbb{R}$ which is smooth and strictly plurisubharmonic on $X \setminus \phi^{-1}(0)$ and such that $\widetilde{\phi} \geq \phi - 1$ on X. In particular, $\widetilde{\phi}$ is an exhaustion function, and by Grauert's Theorem 1.11, X is holomorphically convex. □

Proof of Proposition 1.12. Let $d_C^2 : X \to \mathbb{R}_+$ be the squared distance function to C. This function is convex on X and strongly convex on $X \setminus C$. This is shown in [BH23, Lemma 4.5], and for completeness, we now sketch a proof of the strong convexity of d_C^2 on $X \setminus C$. Fix $\epsilon > 0$ and let $\gamma : [0, L] \to X$ be a geodesic parameterized at unit speed such that $d(\gamma(0), C) \geq \epsilon$ and $d(\gamma(L), C) \geq \epsilon$. Let x and y be the projections of $\gamma(0)$ and $\gamma(L)$ onto C, and let $\eta : [0, L'] \to X$ be the unit-speed geodesic joining x to y. Using [KS93, Corollary 2.1.3 - Formula 2.1(iv)], we get

$$2d_C^2\left(\gamma\left(\frac{L}{2}\right)\right) - d_C^2(\gamma(0)) - d_C^2(\gamma(L)) \leq -\frac{1}{2}(L - L')^2.$$

Finally, since X has sectional curvature bounded above by -1, there exists a positive constant a depending only on ϵ such that $(L - L')^2 \geq aL^2$. Using [GW76, Lemma 1], we obtain that d_C^2 is strongly convex on $X \setminus C$.

To conclude, observe that the function d_C^2 is Γ-invariant, so it defines a convex function $\phi\colon X/\Gamma \to \mathbb{R}$ which is strictly convex outside C/Γ. The proposition then follows from applying Lemma 1.13 to the function ϕ. □

1.1.4 Stein manifolds

The notion of Stein manifold is central to this thesis. To introduce it, we need the following definition, already given as Property (**S**) in the introduction.

Definition 1.14. Let X be a complex manifold. Holomorphic functions on X are said to *separate points* if for any pair of distinct points x and y, there exists a holomorphic function $f\colon X \to \mathbb{C}$ such that $f(x) \neq f(y)$.

A natural obstruction to the fact that holomorphic functions on a complex manifold X separate the points of X is the presence of compact analytic subvarieties of positive dimension. We refer to Proposition 2.7 for examples of criterion preventing the existence of compact analytic subvarieties of positive dimension. We now state the definition of a Stein manifold in the form of a theorem.

Definition and Theorem 1.15. *A complex manifold X is called Stein if it satisfies one of the following equivalent conditions:*

(i) X is holomorphically convex and holomorphic functions on X separate points.

(ii) X is holomorphically convex and does not contain any compact analytic subvariety of positive dimension.

(iii) X admits a smooth strictly plurisubharmonic exhaustion function.

Before givin a sketch of proof of this result, we note that there exists other equivalent characterizations of Stein manifolds. For instance, a complex manifold X is Stein if and only if there exists a proper holomorphic embedding $X \hookrightarrow \mathbb{C}^N$, or if and only if every coherent analytic sheaf on X is acyclic, see [Hör90, Chapters V and VII].

Sketch of proof. The implication $(i) \implies (ii)$ is a consequence of the maximum principle, see Proposition 1.5.
The implication $(ii) \implies (i)$ follows from Remmert's reduction, which states the following. For every holomorphically convex manifold X, there exists a normal variety X' satisfying condition (i) and a proper holomorphic map with connected fibers $f\colon X \to X'$, see for example [Pet94, Theorem 2.1]. In particular, if X does not contain any compact analytic subvariety of positive dimension, the fibers of f reduce to points, so f is a biholomorphism and X satisfies property (i).
The implication $(i) \implies (iii)$ is not very difficult, see for instance [Dem, Theorem I.6.14 and Lemma I.6.17].
The implication $(iii) \implies (i)$ is much more difficult and is called "the solution to the Levi problem". It was proven by Grauert in [Gra58] as a consequence of Theorem 1.9, see also [Hör90] or [Ber10] for other proofs. □

Example. Here is a fundamental class of examples of Stein manifolds, worked out by Greene and Wu [GW79, Proposition 1.17]. The proof uses the solution to the Levi problem.

Let (X, ω) be a complete simply connected Kähler manifold with nonpositive sectional curvature. Then X is a Stein manifold. Indeed, for any point $o \in X$, the function $\phi := d(\cdot, o)^2$ is a smooth exhaustion that is strongly convex and therefore strictly plurisubharmonic by Formula (1.1). This class of examples includes all Hermitian symmetric spaces of noncompact type, including $\mathbb{C}^n$ and the unit ball in $\mathbb{C}^n$.

Remark. A compact manifold of positive dimension never satisfies (**S**) and trivially satisfies (**HC**). There exist noncompact manifolds that admit no non-constant holomorphic functions, for instance a quotient of $\mathbb{C}^2$ by a "generic" rank 3 Abelian subgroup (see [Nap90, Example 4.5] or [AK01]), or a noncompact finite-volume quotient of the unit ball in $\mathbb{C}^n$ whenever $n \geq 2$ (see Proposition **??**). In dimension 1, a theorem of Behnke and Stein states that every noncompact Riemann surface is a Stein manifold [BS47; For81], and consequently, the question of the existence of holomorphic functions on a Riemann surface S depends only on the compactness of S. A (possibly branched) covering of a Stein manifold is Stein [Le 76, Théorème 1]. In general, a fiber bundle with Stein base and Stein fiber is not necessarily a Stein manifold [Sko77; Dem78], although it is under stronger hypotheses on the fiber [Fis70; Siu78].

In Chapter 2, we study quotients of Hermitian symmetric spaces of noncompact type. These quotients are not Stein in general, for instance a compact quotient is never Stein. More elaborate examples of non-Stein quotients are given by (noncompact) finite-volume quotients of the ball in dimension at least 2: we refer to the remark at the end of Section 3.1 for a proof of this fact. We also refer to Section 2.1.2.3 for two examples of parabolic subgroups Γ of $\mathrm{PU}(n,1)$ for which the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not holomorphically convex. One of these examples can be realized as a complex hyperbolic bundle of punctured disks over a *Cousin group*, which is by definition a quotient of an affine space on which all holomorphic functions are constant, see [AK01].

To conclude this section, we state the following version of Docquier–Grauert's theorem, that will be needed later in this thesis.

Theorem 1.16 ([DG60], [Siu78, Theorem 5.2])**.** *Let X be a complex manifold and $f : X \to \mathbb{R}$ be a strictly plurisubharmonic continuous function. Assume that for every real number t, the open subset*

$$X_t := \{x \in X \mid f(x) < t\}$$

of X is a Stein manifold. Then X is a Stein manifold.

1.2 Complex hyperbolic geometry

Among symmetric spaces of noncompact type, complex hyperbolic spaces form the only family which is Hermitian and has rank one. This makes them simply connected Stein manifolds of negative curvature with a biholomorphism group containing interesting discrete subgroups. We first describe the complex hyperbolic space and its biholomorphism group $\mathrm{PU}(n,1)$, then we pass to the study of the discrete subgroups of this Lie group. We begin with the parabolic subgroups and introduce the Busemann functions on the complex hyperbolic space. Then we review the properties of convex-cocompact and geometrically finite subgroups of $\mathrm{PU}(n,1)$. Finally, we introduce the notion of critical exponent of a discrete subgroup as well as Patterson–Sullivan measures.

1.2.1 The complex hyperbolic space and the group $\mathrm{PU}(n,1)$

Our main reference for this section is [Gol99].

1.2.1.1 Construction of the complex hyperbolic space

Let $\mathbb{C}^{n,1}$ denote the complex vector space $\mathbb{C}^{n+1}$ equipped with the following Hermitian form of signature $(n,1)$

$$\langle u,v\rangle := -u_1\overline{v_1} + \sum_{i=2}^{n+1} u_i\overline{v_i},$$

and let q be the quadratic form associated with $\langle\cdot,\cdot\rangle$. Denote by $\xi \in \mathbb{C}^{n,1}\setminus\{0\} \mapsto [\xi] \in \mathbb{CP}^n$ the canonical projection. The *complex hyperbolic space* is defined as

$$\mathbb{H}^n_{\mathbb{C}} = \{[\xi] \in \mathbb{CP}^n \mid q(\xi) < 0\}.$$

As a complex manifold, $\mathbb{H}^n_{\mathbb{C}}$ is biholomorphic to the unit ball in $\mathbb{C}^n$. Indeed, if $[\xi] \in \mathbb{H}^n_{\mathbb{C}}$, then $\xi_1 \neq 0$, so we can write $[\xi] = [1 : z_1, \dots, z_n]$, and the condition $q(\xi) < 0$ rewrites as $\sum_{i=1}^n |z_i|^2 < 1$. The boundary of the complex hyperbolic space is the boundary of $\mathbb{H}^n_{\mathbb{C}}$ viewed as an open subset of $\mathbb{CP}^n$. It is diffeomorphic to a $(2n-1)$-dimensional sphere and is given by

$$\partial\mathbb{H}^n_{\mathbb{C}} := \{[\xi] \in \mathbb{CP}^n \mid q(\xi) = 0\}.$$

The *hyperbolic metric* is a metric on $\mathbb{H}^n_{\mathbb{C}}$ defined as follows: if $p \in \mathbb{C}^{n,1}$ lies on the cone $q(p) = -1$, then the restriction of the Hermitian form to the orthogonal complement $(\mathbb{C}p)^\perp \subset \mathbb{C}^{n,1}$ defines a Hermitian inner product on that space. Since the tangent space to $\mathbb{H}^n_{\mathbb{C}}$ at the point $[p]$ naturally identifies with $(\mathbb{C}p)^\perp$, this yields a Hermitian product on $T_{[p]}\mathbb{H}^n_{\mathbb{C}}$, which can be shown to be independent of the representative p. This metric is Kähler and has negative curvature pinched between -4 and -1 (see [KN96, Theorem IX.7.8]).

A straightforward computation, which we omit (see [Gol99, Theorem 3.1.7]), shows that in the coordinates $(x_1, \dots, x_n) \mapsto [1 : x_1, \dots : x_n]$, the metric has the expression:

$$\langle v,w\rangle_x = \frac{1}{1-\|x\|^2}\left(\langle v,w\rangle_0 + \frac{\langle v,x\rangle_0\langle x,w\rangle_0}{1-\|x\|^2}\right), \tag{1.2}$$

where the product $\langle\cdot,\cdot\rangle_0$ on the right-hand side denotes the standard Hermitian inner product on $T_x\mathbb{H}^n_{\mathbb{C}} \simeq \mathbb{C}^n$.

Explicitly, the hyperbolic distance can be computed using the following formula.

Proposition 1.17 ([Gol99, §3.1.7])**.** *Let $x, y \in \mathbb{H}^n_{\mathbb{C}}$ and let $\widetilde{x}, \widetilde{y} \in \mathbb{C}^{n+1}$ be lifts of x, y. Then:*

$$\cosh^2(d(x,y)) = \frac{\langle \widetilde{x}, \widetilde{y} \rangle \langle \widetilde{y}, \widetilde{x} \rangle}{\langle \widetilde{x}, \widetilde{x} \rangle \langle \widetilde{y}, \widetilde{y} \rangle}. \tag{1.3}$$

The Hermitian products here refer to the Hermitian form on $\mathbb{C}^{n,1}$ with signature $(1, n)$.

1.2.1.2 Holomorphic isometries of the complex hyperbolic space

By construction, the group $\mathrm{U}(n,1)$ of endomorphisms of $\mathbb{C}^{n+1}$ that preserve the Hermitian form $\langle \cdot, \cdot \rangle$ acts by holomorphic isometries on $\mathbb{H}^n_{\mathbb{C}}$, and the homotheties of modulus 1 lie in the kernel of this action. Let $\mathrm{PU}(n,1)$ be the quotient of $\mathrm{U}(n,1)$ by the homotheties of modulus 1. It follows that $\mathrm{PU}(n,1)$ acts by holomorphic isometries on $\mathbb{H}^n_{\mathbb{C}}$. In fact, we have:

Proposition 1.18. *Every biholomorphism of the unit ball is an isometry of $\mathbb{H}^n_{\mathbb{C}}$, and the group of biholomorphic isometries of $\mathbb{H}^n_{\mathbb{C}}$ is $\mathrm{PU}(n,1)$.*

We now describe the classical trichotomy allowing to define the type of an element γ of $\mathrm{PU}(n,1)$ different from the identity. The element γ acts by homeomorphisms on $\overline{\mathbb{H}^n_{\mathbb{C}}}$, and it fixes at least a point in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ by Brouwer's theorem. The element γ is said to be *elliptic* if it fixes a point inside $\mathbb{H}^n_{\mathbb{C}}$, *hyperbolic* (or *loxodromic*) if it does not fix any point inside $\mathbb{H}^n_{\mathbb{C}}$ but fixes two points in $\partial\mathbb{H}^n_{\mathbb{C}}$. In this case, γ is a translation in restriction to the unique geodesic joining these two points. Finally it is *parabolic* if it does not fix any point inside $\mathbb{H}^n_{\mathbb{C}}$ and fixes exactly one point in $\partial\mathbb{H}^n_{\mathbb{C}}$. In this case, γ preserves a foliation called the *horospherical foliation* of $\mathbb{H}^n_{\mathbb{C}}$, see Section 1.2.4. Parabolic elements will be studied in more details in Section 1.2.3.

1.2.1.3 Siegel coordinates and Iwasawa decomposition

Let $\xi_1 = [f_1]$ and $\xi_2 = [f_2]$ be two distinct points in $\partial\mathbb{H}^n_{\mathbb{C}}$, with $f_1, f_2 \in \mathbb{C}^{n,1}$ two isotropic vectors, and set $P = \mathbb{C}f_1 \oplus \mathbb{C}f_2$. Then $P \cap P^\perp = \{0\}$, so that $\mathbb{C}^{n+1} = P \oplus P^\perp$. By renormalizing f_1, we may assume that, in the basis (f_1, f_2), the restriction of $\langle \cdot, \cdot \rangle$ to P has matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Choose an orthonormal basis $(e_3, \ldots, e_{n+1})$ of $P^\perp$, and form the basis $f = (f_1, f_2, e_3, \ldots, e_{n+1})$ of $\mathbb{C}^{n+1}$. In this basis, the quadratic form q associated to the Hermitian form $\langle \cdot, \cdot \rangle$ is given by

$$q(a\xi_1 + b\xi_2 + u) = 2\Re(a\bar{b}) + \|u\|^2.$$

We can then define a global chart of $\mathbb{H}^n_{\mathbb{C}}$ by

$$\begin{cases} \{(a,u) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 2\Re(a) + \|u\|^2 < 0\} & \longrightarrow & \mathbb{H}^n_{\mathbb{C}} \\ (a,u) & \longmapsto & [af_1 + f_2 + u] \end{cases}$$

The coordinates given by the inverse of this global chart will be called *Siegel coordinates*.

The next lemma provides a diffeomorphism between $\mathbb{H}^n_{\mathbb{C}}$ and the product of two subgroups A and N of $\mathrm{PU}(n,1)$. It is a direct consequence of the "Iwasawa decomposition" of $\mathrm{PU}(n,1)$, but we will rather make an explicit computation that will also be used to give an expression for the Busemann functions of $\mathbb{H}^n_{\mathbb{C}}$ in Section 1.2.4. We first introduce the two subgroups A and N.

For every real number t, we denote by a_t the element of $\mathrm{PU}(n,1)$ whose matrix in the basis f is given by

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & \mathrm{I}_{n-2} \end{pmatrix}, \tag{1.4}$$

and we define A as the group of all the elements a_t with $t \in \mathbb{R}$.

Next, for every couple $(b,c) \in \mathbb{C}^{n-1} \times \mathbb{R}$, we denote by $n_{(b,c)}$ the element of $\mathrm{PU}(n,1)$ whose matrix in the basis f is given by

$$\begin{pmatrix} 1 & -\frac{\|b\|^2}{2} + ic & -{}^t\bar{b} \\ 0 & 1 & 0 \\ 0 & b & \mathrm{I}_{n-2} \end{pmatrix}, \tag{1.5}$$

and we define N as the group of all the elements $n_{(b,c)}$ with $(b,c) \in \mathbb{C}^{n-1} \times \mathbb{R}$. In Chapter 3, the group N will be called N_ξ to emphasize the point it stabilizes.

We have now introduced enough explicit elements of $\mathrm{PU}(n,1)$ to state the lemma.

Lemma 1.19. *Let $x \in \mathbb{H}^n_{\mathbb{C}}$ with Siegel coordinates (a,u) and $o = [-f_1 + f_2]$ be the point of Siegel coordinates $(-1,0)$. Then there is a unique element $(a_t, n_{(b,c)}) \in A \times N$ such that $x = a_t n_{(b,c)} \cdot o$. Explicitely, t, b and c are given by $t = \frac{1}{2}\log\left(\frac{-2}{2\Re(a)+\|u\|^2}\right)$, $b = e^{-t}u$ and $c = e^{-2t}\Im(a)$.*

Proof. This is a consequence of the following computation.

$$\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} 1 & -\frac{\|b\|^2}{2} + ic & -{}^t\bar{b} \\ 0 & 1 & \\ 0 & b & I_{n-2} \end{pmatrix} \cdot [f_2 - f_1] = \left[e^{2t}\left(-\frac{\|b\|^2}{2} - 1 + ic\right) f_1 + f_2 + e^t b \right].$$

□

1.2.2 Discrete subgroups of $\mathrm{PU}(n,1)$

In this section, we recall some classical facts about discrete subgroups of $\mathrm{PU}(n,1)$. Our main references for this section are [CNS13], [Kap01] and [KL18].

Let Γ be a subgroup of $\mathrm{PU}(n,1)$. Then Γ acts properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$ if and only if Γ is discrete. Moreover, a discrete subgroup Γ acts freely on $\mathbb{H}^n_{\mathbb{C}}$ if and only if it is torsion-free. For the reminder of this subsection 1.2.2, we fix a discrete and torsion-free subgroup Γ of $\mathrm{PU}(n,1)$ and a point $o \in \mathbb{H}^n_{\mathbb{C}}$. Then the orbit $\Gamma \cdot o$ does not accumulate in $\mathbb{H}^n_{\mathbb{C}}$ but has at least one accumulation point in the compact set $\overline{\mathbb{H}^n_{\mathbb{C}}}$. The set of accumulation points of $\Gamma \cdot o$ in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ therefore equals $\overline{\Gamma o} \cap \partial\mathbb{H}^n_{\mathbb{C}}$.

Definition 1.20. The set of accumulation points of an orbit $\Gamma \cdot o$ in $\partial\mathbb{H}^n_{\mathbb{C}}$ is called the *limit set* of Γ and it is denoted by $\Lambda(\Gamma)$. It does not depend on the choice of $o \in \mathbb{H}^n_{\mathbb{C}}$.

Definition 1.21. We say that Γ is *nonelementary* if its limit set $\Lambda(\Gamma)$ is infinite.

It can be shown (see [Kap01, §3.6]) that $\Lambda(\Gamma)$ contains 0, 1, 2, or infinitely many elements. Consequently, Γ is elementary if and only if its limit set contains at most two elements. Moreover, the group generated by a loxodromic and a parabolic element sharing a fixed point is never discrete. This can be seen as an easy consequence of [KL18, Exercice 4.30]. Given a discrete subgroup Γ of $\mathrm{PU}(n,1)$, we say that a point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ is parabolic (resp. loxodromic) if it is the fixed point of a parabolic (resp. loxodromic) element of Γ.

For any point $\xi \in \Lambda(\Gamma)$ is the fixed point of either a hyperbolic or a parabolic element of Γ. Thus discrete and torsion-free elementary subgroups of $\mathrm{PU}(n,1)$ fall into two types. Either Γ acts by translation along a fixed geodesic, and is virtually cyclic. In this case, Γ is called hyperbolic (or loxodromic), and it is easy to see that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold (this can be seen for instance as a consequence of Proposition 1.12, see also [dFab98]). The most interesting case is when Γ is parabolic, that is when all elements of Γ fix a unique point of $\partial\mathbb{H}^n_{\mathbb{C}}$. The stabilizer of a point of $\partial\mathbb{H}^n_{\mathbb{C}}$ will be studied in more details in Section 1.2.3.

We now come back to the nonelementary case, and we refer to [Kap01, Corollary 3.26] for the proof of the following lemma.

Lemma 1.22. *Assume that Γ is nonelementary. Then $\Lambda(\Gamma)$ is the closure of the attracting fixed points of the hyperbolic elements of Γ.*

From this lemma, we deduce the following characterization of the limit set, which we will use later.

Corollary 1.23. *Assume that Γ is nonelementary. Then $\Lambda(\Gamma)$ is the smallest non-empty closed subset of $\partial\mathbb{H}^n_{\mathbb{C}}$ invariant under Γ.*

Proof. By definition, $\Lambda(\Gamma)$ is non-empty, closed, and Γ-invariant. Let F be a non-empty closed Γ-invariant set, and let $\xi \in F$. We show that F contains all attracting fixed points of hyperbolic elements of Γ. Let γ be a hyperbolic element with attracting fixed point z and repelling fixed point w. There exists $f \in \Gamma$ such that $f\xi \neq w$, hence $\gamma^n f\xi \to z$, which implies $z \in F$ and therefore $\Lambda(\Gamma) \subset F$. ☐

Definition 1.24. Assume that Γ is nonelementary. The convex hull $\mathrm{Conv}(\Lambda(\Gamma))$ of its limit set $\Lambda(\Gamma)$ is the intersection of all the convex subsets of $\mathbb{H}^n_{\mathbb{C}}$ whose closure in $\overline{\mathbb{H}^n_{\mathbb{C}}}$ contain $\Lambda(\Gamma)$. The convex core $C(\Gamma)$ is the quotient of $\mathrm{Conv}(\Lambda(\Gamma))$ inside $\mathbb{H}^n_{\mathbb{C}}/\Gamma$.

In general, it is difficult to construct convex sets in variable curvature, and the geometry of the convex core is not very well known. Anderson managed to construct enough convex sets to show that for any discrete subgroup Γ, we have :

$$\partial\,\mathrm{Conv}(\Lambda(\Gamma)) = \Lambda(\Gamma),$$

see [And83].

Definition 1.25. We say that $x \in \partial\mathbb{H}^n_{\mathbb{C}}$ is a *discontinuity point* of Γ if there exists an open neighborhood U of x such that the set $\{\gamma \in \Gamma \mid U \cap \gamma U \neq \emptyset\}$ is finite. The set $\Omega(\Gamma)$ of discontinuity points of Γ is called its *domain of discontinuity*.

The next proposition links the limit set and the domain of discontinuity of a discrete subgroup of $\mathrm{PU}(n,1)$. We refer to [CNS13, Theorem 1.2.19] for a proof in the case of discrete subgroups of $\mathrm{PO}(n,1)$, that also works *mutatis mutandis* for discrete subgroups of $\mathrm{PU}(n,1)$.

Proposition 1.26. *We have the equality $\Lambda(\Gamma) = \partial\mathbb{H}^n_{\mathbb{C}} \setminus \Omega(\Gamma)$. Moreover, Γ acts properly discontinuously on $\mathbb{H}^n_{\mathbb{C}} \cup \Omega(\Gamma)$.*

1.2.3 Parabolic subgroups of $\mathrm{PU}(n,1)$

We now describe more precisely the structure of parabolic subgroups. Let us fix as in Section 1.2.1.3 a basis $f = (f_1, f_2, e_3, \ldots, e_{n+1})$. Then the stabilizer of the point $\xi = [f_1] \in \partial\mathbb{H}^n_{\mathbb{C}}$ in $\mathrm{PU}(n,1)$ decomposes as

$$\mathrm{Stab}_\xi(\mathrm{PU}(n,1)) = MAN,$$

where A and N have been defined in Section 1.2.1.3 and M is the subgroup of $\mathrm{PU}(n,1)$ consisting of all elements whose matrix in the basis f is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & T \end{pmatrix},$$

with $T \in \mathrm{U}(n-1)$. In particular, N is the *unipotent radical* of $\mathrm{Stab}_\xi(\mathrm{PU}(n,1))$ and it depends only on $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$.

For $T \in \mathrm{U}(n-1), b \in \mathbb{C}^{n-1}$ and $c \in \mathbb{R}$, let (T,b,c) denote the element of the group MN defined in the basis f by the matrix

$$\begin{pmatrix} 1 & -\frac{\|b\|^2}{2} + ic & -\langle T\cdot, b\rangle \\ 0 & 1 & 0 \\ 0 & b & T \end{pmatrix}.$$

The group law on MN is given by

$$(T,b,c)\cdot(T',b',c') = (TT', b+Tb', c+c'+\Im\langle b, Tb'\rangle), \tag{1.6}$$

where $\langle\cdot,\cdot\rangle$ is the standard Hermitian product on $\mathbb{C}^{n-1}$. This group identifies with a semi-direct product $\mathrm{U}(n-1)\ltimes N$, and N is isomorphic to the *Heisenberg group* of real dimension $2n-1$. The center $Z(N)$ of N is the set of elements of the form $(\mathrm{Id}, 0, c)$, with $c \in \mathbb{R}$. We denote by π the projection of $\mathrm{U}(n-1)\ltimes N$ onto $\mathrm{U}(n-1)$, and by Π the morphism

$$\mathrm{U}(n-1)\ltimes N \to \mathrm{U}(n-1)\ltimes \mathbb{C}^{n-1}$$

which to $(T,b,c) \in \mathrm{U}(n-1)\ltimes N$ associates the holomorphic isometry $z \mapsto Tz+b$ of $\mathbb{C}^{n-1}$. For all $\gamma \in MN$, we write

$$\gamma = (\pi(\gamma), b(\gamma), c(\gamma)), \text{ with } \pi(\gamma)\in \mathrm{U}(n-1), b(\gamma)\in\mathbb{C}^{n-1}, c(\gamma)\in\mathbb{R}, \text{ and}$$
$$\Pi(\gamma) := (\pi(\gamma), b(\gamma)).$$

A discrete parabolic subgroup Γ of $\mathrm{PU}(n,1)$ fixing $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ identifies with a discrete subgroup of $\mathrm{U}(n-1)\ltimes N$. It is said to be *unipotent* if $\pi(\Gamma)$ is trivial.

We will need later the following result, which is presumably classical, and we include it for completeness.

Lemma 1.27. *Let Γ be a discrete and torsion-free parabolic subgroup of $\mathrm{PU}(n,1)$. Then there exists a finite index subgroup Γ_1 of Γ such that $\Pi(\Gamma_1)$ is Abelian.*

Proof. As before, let us fix a basis of $\mathbb{C}^{n+1}$ which induces an identification between Γ and a discrete subgroup of $\mathrm{U}(n-1)\ltimes N$. By Margulis lemma (see Theorem 1.37 or [Bow95]), Γ is virtually nilpotent. The existence of Γ_1 follows from the classical fact that a nilpotent subgroup of $U(n-1) \ltimes \mathbb{C}^{n-1}$ is virtually Abelian. To show this fact, the first observation, that we will not prove here, is that a nilpotent subgroup of $U(n-1)$ is virtually Abelian. Let Γ_1 be a finite index nilpotent subgroup of Γ such that $\pi(\Gamma_1)$ is Abelian. Seeking a contradiction, let us assume that the nilpotent group $\Pi(\Gamma_1)$ is not Abelian. There is a non-trivial element z in the center of $\Pi(\Gamma_1)$ which can be written as a product of commutators $z = [x_1, y_1] \dots [x_k, y_k]$, with $x_1, \dots, x_k, y_1, \dots, y_k \in \Pi(\Gamma_1)$. Let us write

$$\begin{aligned} x_i &= (\pi(x_i), b(x_i)), \\ y_i &= (\pi(y_i), b(y_i)), \\ z &= (\mathrm{Id}, b(z)), \end{aligned}$$

with $\pi(x_i), \pi(y_i) \in U(n-1)$ which commute, and $b(x_i), b(y_i), b(z) \in \mathbb{C}^{n-1}$. We will now show that $b(z) = 0$, which means that z is trivial, a contradiction. Given $\pi_1, \pi_2 \in U(n-1)$ which commute and $b_1, b_2 \in \mathbb{C}^{n-1}$, we compute that

$$\begin{aligned} [(\pi_1, b_1), (\pi_2, b_2)] &= (\pi_1, b_1)\ (\pi_2, b_2)\ (\pi_1^{-1}, -\pi_1^{-1} b_1)\ (\pi_2^{-1}, -\pi_2^{-1} b_2) \\ &= (\pi_1 \pi_2, b_1 + \pi_1 b_2)(\pi_1^{-1} \pi_2^{-1}, -\pi_1^{-1} b_1 - \pi_1^{-1} \pi_2^{-1} b_2) \\ &= (\mathrm{Id}, b_1 + \pi_1 b_2 + (\pi_1 \pi_2)(-\pi_1^{-1} b_1 - \pi_1^{-1} \pi_2^{-1} b_2)) \\ &= (\mathrm{Id}, (\mathrm{Id} - \pi_2) b_1 - (\mathrm{Id} - \pi_1) b_2). \end{aligned}$$

Therefore

$$b(z) = \sum_{i=1}^{k} (\mathrm{Id} - \pi(y_i)) b(x_i) - (\mathrm{Id} - \pi(x_i)) b(y_i). \tag{1.7}$$

Moreover, z commutes with all elements of $E := \{x_1, \dots, x_k, y_1, \dots, y_k\}$, which implies that

$$\forall \gamma \in E, \quad \pi(\gamma) b(z) = b(z). \tag{1.8}$$

Choose a basis $e = (e_1, \dots, e_{n-1})$ that diagonalizes all elements of $\pi(E)$ and express $\pi(\gamma)$ in this basis as $\mathrm{Diag}(a_1(\gamma), \dots, a_{n-1}(\gamma))$ for $\gamma \in E$. For all $j \in \{1, \dots, n-1\}$, if there exists $\gamma \in E$ such that $a_j(\gamma) \neq 1$, then the j^{th} coordinate $b_j(z)$ of $b(z)$ in the basis e must vanish according to Formula (1.8), and if $a_j(\gamma) = 1$ for all $\gamma \in E$, then $b_j(z) = 0$ according to Formula (1.7). Thus $b(z) = 0$, which gives the contradiction we were looking for and proves that $\Pi(\Gamma_1)$ is Abelian. □

1.2.4 Busemann Functions

We now describe *Busemann functions* which are a classical object of study on manifolds of nonpositive curvature. Let $\xi \in \partial \mathbb{H}^n_{\mathbb{C}}$ and γ be a geodesic such that $\lim_{t \to \infty} \gamma(t) = \xi$. For every $x \in \mathbb{H}^n_{\mathbb{C}}$, the function $t \mapsto d(x, \gamma(t)) - t$ is decreasing and bounded below, hence converges to a limit $B_\gamma(x) \in \mathbb{R}$.

Definition 1.28. Let $\xi \in \partial \mathbb{H}^n_{\mathbb{C}}$, and $x, y \in \mathbb{H}^n_{\mathbb{C}}$. The Busemann function at ξ is defined by $B_\xi(x, y) := B_\gamma(x)$ where γ is the unique geodesic such that $\gamma(0) = y$ and $\gamma(t) \to \xi$ as $t \to +\infty$.

The Busemann function satisfies the cocycle relation:

$$B_\xi(x, z) = B_\xi(x, y) + B_\xi(y, z),$$

for every $x, y, z \in \mathbb{H}^n_{\mathbb{C}}$, see for instance [Qui06, Section 3.2]. Thus for all $x, y \in \mathbb{H}^n_{\mathbb{C}}$, the two functions $z \mapsto B_\xi(z, x)$ and $z \mapsto B_\xi(z, y)$ differ only by a constant. In particular, they share the same level sets and sublevel sets. We then define:

Definition 1.29. A *horoball* (resp. *horosphere*) in $\mathbb{H}^n_{\mathbb{C}}$ centered at ξ is a sublevel set (resp. a level set) of a Busemann function $z \mapsto B_\xi(z, x)$.

Lemma 1.30. *In the Siegel coordinates (a, u) introduced above, the Busemann functions at the point $\xi = [f_1]$ are translates of the function B_ξ defined by*

$$B_\xi(a, u) = \frac{1}{2} \log \left(\frac{-2}{2\Re(a) + \|u\|^2} \right).$$

Proof. This is a direct consequence of Lemma 1.19 together with [Qui06, Lemma 3.8]. □

As a consequence, one can prove that parabolic elements at ξ preserve the Busemann function at ξ and the horospherical foliation of $\mathbb{H}^n_{\mathbb{C}}$ defined by its level sets. Moreover, a computation shows that

$$i\partial\bar{\partial} B_\xi = \omega, \tag{1.9}$$

and in particular the Busemann function B_ξ is strictly plurisubharmonic on $\mathbb{H}^n_{\mathbb{C}}$. Since the action of $\mathrm{PU}(n, 1)$ on $\partial\mathbb{H}^n_{\mathbb{C}}$ is transitive, we deduce that this formula holds true for every $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$.

1.2.5 Convex-Cocompact subgroups of $\mathrm{PU}(n, 1)$

We now define the class of convex-cocompact groups.

Definition 1.31. A discrete and nonelementary subgroup Γ of $\mathrm{PU}(n, 1)$ is said to be *convex-cocompact* if there exists a geodesically convex subset of $\mathbb{H}^n_{\mathbb{C}}$ on which Γ acts cocompactly. Equivalently, Γ is convex-cocompact if and only if it acts cocompactly on the convex hull of its limit set.

We will give some equivalent characterization of convex-cocompact groups. We first need the definition of a conical point of the limit set. This definition makes sense in a much larger framework, for instance a complete simply connected Riemannian manifold of nonpositive curvature, or even a CAT(-1) space.

Definition 1.32. Let Γ be a discrete subgroup of $\mathrm{PU}(n, 1)$ and fix a basepoint $o \in \mathbb{H}^n_{\mathbb{C}}$. A point $\xi \in \Lambda(\Gamma)$ is called conical if it is the limit of a sequence $(\gamma_n \cdot o)_n$ that stays at bounded distance of a(ny) geodesic ray pointing to ξ.

There are different characterizations for convex-cocompact subgroups of $\mathrm{PU}(n, 1)$. We give the main ones in the following theorem.

Theorem 1.33 ([KL18])**.** *Let Γ be a discrete and nonelementary subgroup of $\mathrm{PU}(n, 1)$. The following are equivalent:*

1. *Γ is convex-cocompact.*
2. *The action of Γ on $\mathbb{H}^n_{\mathbb{C}} \cup \Omega(\Gamma)$ is cocompact.*
3. *Every point $\xi \in \Lambda(\Gamma)$ is conical.*

Examples. Here are (classes of) examples of convex-cocompact subgroups of $\mathrm{PU}(n, 1)$.

1. Every cocompact lattice.
2. Every Schottky group (see [CNS13, Section 1.2.5] or [KL18, Example 4.3.6]).

3. Every complex Fuchsian group. By definition, these are groups that act cocompactly on a holomorphically embedded totally geodesic copy of a disk inside $\mathbb{H}^n_{\mathbb{C}}$.

4. Every real Fuchsian group. By definition, these are groups that act cocompactly on a totally real and totally geodesic copy of a disk inside $\mathbb{H}^n_{\mathbb{C}}$.

5. Let $g \geq 2$ and $\rho\colon \pi_1(\Sigma_g) \to \mathrm{PU}(n,1)$ be a representation of the fundamental group of the genus g surface Σ_g into $\mathrm{PU}(n,1)$. To such a representation, one can associate a number $\tau \in \frac{2}{n+1}\mathbb{Z}$ called the Toledo invariant. In the case where ρ is discrete and faithful, it is the real number
$$\tau := \frac{1}{2\pi}[\phi^*\omega] \in H^2(\Sigma_g, \mathbb{R}) \simeq \mathbb{R},$$
where $\phi\colon \Sigma_g \to \mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ is any homotopy equivalence between Σ_g and $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$. We refer to [BI07; Bur+05] for a much more general definition. It satisfies a Milnor-Wood inequality
$$|\tau| \leq 2g - 2. \tag{1.10}$$
Moreover, $|\tau|$ is maximal if and only if $\rho(\Gamma)$ is a complex Fuchsian group, see [Tol89] and it vanishes if and only if $\rho(\Gamma)$ is a deformation of a real Fuchsian group, see [Xia03]. It is proven in [GKL01] that for every integer τ satisfying Inequality (1.10) there is a faithful and discrete representation $\rho\colon \Sigma_g \to \mathrm{PU}(n,1)$ whose Toledo invariant is τ. As mentioned in the introduction, the article [Bro25] construct examples of convex-cocompact representations $\rho\colon \pi(\Sigma_g) \to \mathrm{PU}(2,1)$ with intermediate Toledo number and whose quotient $\mathbb{H}^n_{\mathbb{C}}/\rho(\Gamma)$ contains a compact analytic subvariety of positive dimension.

6. Every small perturbation of a convex-cocompact group remains convex-cocompact, see for instance [Bow98, Proposition 4.1].

7. Granier constructed an example of a convex-cocompact subgroup of $\mathrm{PU}(2,1)$ whose limit set is homeomorphic to the Menger curve [Gra15].

1.2.6 Geometrically finite subgroups of $\mathrm{PU}(n,1)$

We define here the class of *geometrically finite* subgroups of $\mathrm{PU}(n,1)$, that contains the convex-cocompact subgroups. In general, a geometrically finite group contains parabolic elements. For this section, our main reference is [Bow95].

1.2.6.1 Definition

We begin with the definition of a bounded parabolic point. We recall that a parabolic point of a discrete subgroup $\Gamma < \mathrm{PU}(n,1)$ is the fixed point of a parabolic element of Γ.

Definition 1.34. Let Γ be a discrete subgroup of $\mathrm{PU}(n,1)$. A parabolic point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ of Γ is called *bounded* if the action of the parabolic subgroup $\mathrm{Stab}_\Gamma(\xi)$ on $\Lambda(\Gamma) \setminus \{\xi\}$ is cocompact.

The following definition of a geometrically finite group uses the notion of conical points that has been defined in Definition 1.32.

Definition 1.35. A nonelementary discrete subgroup Γ of $\mathrm{PU}(n,1)$ is called *geometrically finite* if every element $\xi \in \Lambda(\Gamma)$ is either a bounded parabolic or a conical point.

Examples of geometrically finite subgroups of $\mathrm{PU}(n,1)$ include convex-cocompact subgroups, non-uniform lattices, but also the examples constructed in [GP92] and [FP03]. They also include examples from the rich theory of geometrically finite subgroups of $\mathrm{PO}(n,1) \subset \mathrm{PU}(n,1)$, see for instance [Mas87, Chapter VI].

1.2.6.2 Structure of a geometrically finite quotient of the ball

To understand the structure of a geometrically finite quotient of $\mathrm{PU}(n,1)$, we need the notion of thick-thin decomposition that was developed by Margulis.

Definition 1.36. Let (M,g) be a complete Riemannian manifold and fix $\epsilon > 0$. The ϵ-thick part of M, denoted by $\mathrm{thick}_\epsilon(M)$, is the set of all points $x \in M$ for which every loop based at x of length smaller than ϵ is contractible. The ϵ-thin part of M, denoted by $\mathrm{thin}_\epsilon(M)$, is the complement of $\mathrm{thick}_\epsilon(M)$.

From this definition, it is easily seen that $x \in \mathrm{thin}_\epsilon(M)$ if and only if

$$\exists \gamma \in \pi_1(M,x) \setminus \{\mathrm{Id}\}, \quad d(x,\gamma x) \leq \epsilon,$$

and thus $x \in \mathrm{thin}_\epsilon(M)$ if and only if the injectivity radius of M at x is smaller than $\epsilon/2$.

When M has pinched sectional curvature, the structure of the thin part of M is understood thanks to the Margulis lemma.

Theorem 1.37 (Margulis lemma, [BGS85])**.** *Let M be a complete Riemannian manifold of sectional curvature pinched between $-K$ and 0 for some positive real number K. There exists a constant ϵ_0 such that for every $x \in \mathrm{thin}_{\epsilon_0}(M)$ the group generated by the elements of $\pi_1(M,x)$ such that $d(x,\gamma x) \leq \epsilon_0$ is virtually nilpotent. The constant ϵ_0 depends only on the dimension of M and on K.*

The following theorem provides the description of geometrically finite quotients of $\mathbb{H}^n_{\mathbb{C}}$ that will be used in the proof of Theorem A. Recall that the convex-core of Γ and its domain of discontinuity $\Omega(\Gamma)$ are defined in Definitions 1.24 and 1.25, respectively.

Theorem 1.38. *[Bow95] Let Γ be a nonelementary discrete subgroup of $\mathrm{PU}(n,1)$. Then Γ is geometrically finite if and only if for a(ny) positive ϵ less than the Margulis constant ϵ_0, the intersection of the convex core of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ and the ϵ-thick part of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is compact.*

Moreover, in this case, the ϵ-thick part Q of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is relatively compact in $(\mathbb{H}^n_{\mathbb{C}} \cup \Omega(\Gamma))/\Gamma$, and the thin part of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ consists of a finite number of parabolic ends, meaning that $X_\Gamma := \mathbb{H}^n_{\mathbb{C}}/\Gamma$ decomposes as

$$X_\Gamma =: Q \cup \bigcup_{i=1}^{k} E_i, \tag{1.11}$$

where k is an integer, and for $i \in \{1,\dots,k\}$, each E_i is an open subset of X_Γ biholomorphic to the quotient of a horoball $B_i^{-1}((-\infty,0))$ by a maximal parabolic subgroup P_i of Γ, for some Busemann function B_i.

1.2.7 Critical exponent of a discrete subgroup of $\mathrm{PU}(n,1)$

The critical exponent of a discrete subgroup Γ of $\mathrm{PU}(n,1)$ has already been defined in the introduction; we recall it here for the reader's convenience. It measures the growth rate of Γ-orbits, and can be seen as a discrete version of the growth rate of ball volumes on $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. The definition makes sense in more generality, for instance if X is a complete simply connected Riemannian manifold of curvature bounded above by -1 and Γ is a group acting properly discontinuously by isometries on X.

Definition 1.39. Let Γ be a discrete subgroup of $\mathrm{PU}(n,1)$. The *critical exponent* of Γ is the real number $\delta(\Gamma)$ defined by

$$\delta(\Gamma) = \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty\},$$

where o is an arbitrary point in $\mathbb{H}^n_{\mathbb{C}}$. It does not depend on the choice of o.

An equivalent definition is

$$\delta(\Gamma) = \limsup_{T\to\infty} \frac{\log \#\{\gamma \in \Gamma \mid d(o,\gamma o) \leq T\}}{T}.$$

Here are some examples and facts about the critical exponent of discrete subgroups of $\mathrm{PU}(n,1)$.

Examples.

- Let Γ be an elementary discrete subgroup of $\mathrm{PU}(n,1)$. If Γ is hyperbolic, it is easy to see that $\delta(\Gamma) = 0$, because
$$\forall s > 0, \sum_{k\in\mathbb{Z}} e^{-s|k|\ell} < +\infty,$$
where ℓ is the translation length along the geodesic preserved by Γ. On the contrary, nontrivial parabolic subgroups always have positive critical exponent (see Formula (1.12) below).

- Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{PO}(2,1)$ that acts on $\mathrm{PU}(n,1)$ by preserving a totally geodesic and totally real copy of the disk. Then $\delta(\Gamma) = 1$. Moreover, any small deformation of Γ in $\mathrm{PU}(n,1)$ has critical exponent bigger than 1, see [Bow79, Theorem 2].

- Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R}) \simeq \mathrm{PU}(1,1)$ that acts on $\mathrm{PU}(n,1)$ by preserving a totally geodesic and holomorphic copy of the disk. Then $\delta(\Gamma) = 2$.

- The critical exponent of a lattice in $\mathrm{PU}(n,1)$ is $2n$. This is the maximum value possibly achieved by critical exponents of subgroups of $\mathrm{PU}(n,1)$. There is no gap phenomenon : there are discrete subgroups of $\mathrm{PU}(n,1)$ with critical exponent arbitrarily close to $2n$, see [DL24].

There is also an interpretation of the critical exponent in terms of the limit set, that we now describe succintly, although we will not use it in the sequel. Denote by $\Lambda_c(\Gamma)$ the conical limit set, which consists of all conical limit points (defined in Section 1.2.5). Then the critical exponent of Γ is the Hausdorff dimension of $\Lambda_c(\Gamma)$ for the sub-Riemannian metric associated with the CR-structure on $\partial\mathbb{H}^n_{\mathbb{C}}$, see [Cor90; CI99]. This generalizes a classical result for discrete groups of $\mathrm{PO}(n,1)$ acting on the real hyperbolic space, and in this setting the Hausdorff dimension of the conical limit set is computed with respect to a(ny) Riemannian metric on the sphere, see for instance [Sul79].

We now describe a formula for the critical exponent of a discrete and torsion-free parabolic subgroup Γ of $\mathrm{PU}(n,1)$, for which we refer to [CI99] or [DOP00, §3]. We will use the notation of Section 1.2.3. Let Γ_1 be a finite index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian, given by Lemma 1.27. Define $l \in \{0,1\}$ as the dimension of the real subspace spanned by $Z(N) \cap \Gamma_1$, where $Z(N) \simeq \mathbb{R}$ is the center of N, and $k \in \{0, \dots 2n-2\}$ as the dimension of the subspace of $\mathbb{C}^{n-1}$ spanned by $\{b(\gamma) \mid \gamma \in \Gamma_1\}$. Then

$$\delta(\Gamma) = \frac{2l+k}{2}. \tag{1.12}$$

1.2.8 Patterson–Sullivan measures

Let Γ be a discrete subgroup of $\mathrm{PU}(n,1)$ and $\delta = \delta(\Gamma)$ be its critical exponent. A *Patterson–Sullivan measure* is a Γ-conformal density of dimension δ, that is, a family of measures $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ on $\partial \mathbb{H}^n_{\mathbb{C}}$ such that $\gamma_* \mu_x = \mu_{\gamma x}$ for all $x \in \mathbb{H}^n_{\mathbb{C}}$ and $\gamma \in \Gamma$, and such that

$$\forall x, y \in \mathbb{H}^n_{\mathbb{C}}, \quad \frac{d\mu_x}{d\mu_y} = e^{-\delta(\Gamma) B_\bullet(x,y)}, \tag{1.13}$$

where $B_\bullet(x,y)$ is the Busemann function defined in Subsection 1.2.4.

Patterson and Sullivan showed that such a measure always exists provided Γ is nonelementary. We recall their construction when Γ is of divergent type, that is when the series

$$\sum_{\gamma \in \Gamma} e^{-\delta d(o, \gamma o)}$$

diverges. The general case is explained for example in [Nic89, Section 3.1]. Fix an arbitrary point $o \in \mathbb{H}^n_{\mathbb{C}}$ and, for $s > \delta$, define

$$\Phi(s) := \sum_{\gamma \in \Gamma} e^{-s d(o, \gamma o)}.$$

The space of probability measures on $\overline{X} := X \cup \partial X$ being compact for the weak topology, we define μ_o as an accumulation point, when $s > \delta$ tends to δ, of the probability measures

$$\frac{1}{\Phi(s)} \sum_{\gamma \in \Gamma} e^{-s d(o, \gamma o)} \delta_{\gamma o},$$

where $\delta_{\gamma o}$ denotes the Dirac measure at γo. The other measures μ_x for $x \in X$ are defined from μ_o via Formula (1.13). It is easily seen that all these measures are finite and supported on the limit set $\Lambda(\Gamma)$ of Γ. In fact, we get more precisely the following result, as a consequence of Corollary 1.23.

Lemma 1.40. *Let Γ be a nonelementary subgroup of* $\mathrm{PU}(n,1)$*, and $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ a Patterson-Sullivan measure associated with Γ. Then for all $x \in \mathbb{H}^n_{\mathbb{C}}$, the support of μ_x is exactly $\Lambda(\Gamma)$.*

Remark. Patterson–Sullivan theory actually extends to the larger setting of complete simply connected Riemannian manifolds of curvature bounded above by -1. Let X be such a manifold, and Γ a group that acts freely and properly discontinuously by isometries on X. We denote by ∂X the visual boundary of X, defined as the quotient of the set of geodesic rays in X by the equivalence relation “remaining at bounded distance”. For every $\xi \in \partial X$, one can define a Busemann function $(x,y) \mapsto B_\xi(x,y)$ centered at ξ as in Section 1.2.4. An important fact, proved by Heintze and Hof in [HH77, Proposition 3.1] is that Busemann functions are of class $\mathcal{C}^2$. Finally, assume that Γ is *nonelementary*, in the sense that its limit set, defined as in Defition 1.20, is infinite. Then a Patterson–Sullivan measure associated with Γ is a family of finite measures $(\mu_x)_{x\in X}$ defined on ∂X, such that $\gamma_*\mu_x = \mu_{\gamma x}$ for all $\gamma \in \Gamma$ and $x \in X$, and such that

$$\forall x, y \in X, \quad \frac{d\mu_x}{d\mu_y} = e^{-\delta(\Gamma)B_\bullet(x,y)}.$$

The above construction works in this larger setting and shows the existence of a Patterson–Sullivan measure on ∂X associated with Γ.

1.3 Higher rank Hermitian symmetric spaces

In this section, we describe the features of Hermitian symmetric spaces that we need for the proofs of Theorems D and Proposition F. The first three subsections are an account of very classical notions, namely root space decomposition, Cartan projection, visual boundary and parabolic subgroups. The main references for these subsections are [Kna96] or [Hel01]. The fourth subsection is specific to Hermitian symmetric spaces and describe some already known results about the compatibility between the root space decomposition and the complex structure. Finally, we introduce Busemann functions in higher rank, describe some results about Patterson–Sullivan measures in higher rank and introduce the growth indicator function of Quint together with the notion of entropy.

1.3.1 Root space of a symmetric space of noncompact type

Let G be a simple Lie group of noncompact type with Lie algebra denoted by $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the eigenspace decomposition of $\mathfrak{g}$ with respect to a Cartan involution θ of $\mathfrak{g}$. Namely, $\mathfrak{k} = \ker(\theta - \mathrm{Id})$ and $\mathfrak{p} = \ker(\theta + \mathrm{Id})$. The symmetric space associated to G is the simply connected Riemannian manifold $\mathbb{X} := G/K$, equipped with the complete metric induced by (a multiple of) the Killing form. At the point $o = eK$, its tangent space identifies naturally with $\mathfrak{p}$. The sectional curvature $\mathcal{K}$ at this point is given by the following expression: let (v, w) be an orthonormal basis of a plane $\Pi \subset T_o\mathbb{X} \simeq \mathfrak{p}$. Then

$$\mathcal{K}(\Pi) = -\langle \mathrm{ad}_v^2(w), w\rangle = -\|[v, w]\|^2.$$

In particular, $\mathbb{X}$ has nonpositive sectional curvature.

Let $\mathfrak{a}$ be a maximal Abelian subalgebra contained in $\mathfrak{p}$. All these subalgebras are conjugate under the adjoint action of K and their common dimension is called the *real rank* of G, see [Kna96, Theorem VI.6.51]. For any linear form $\lambda \in \mathfrak{a}^*$, define

$$\mathfrak{g}^\lambda := \{v \in \mathfrak{g} \mid \forall h \in \mathfrak{a},\ ad(h)v = \lambda(h)v\}.$$

Call λ a *(restricted) root* of $(\mathfrak{g}, \mathfrak{a})$ if $\lambda \neq 0$ and $\mathfrak{g}^\lambda \neq \{0\}$. Denoting by Λ the finite set of roots, we get a decomposition

$$\mathfrak{g} = \mathfrak{g}^0 \oplus \bigoplus_{\lambda\in\Lambda} \mathfrak{g}^\lambda,$$

see [Kna96, Proposition VI.6.40].

The *visual boundary* $\partial_{\mathrm{vis}}\mathbb{X}$ is defined as the quotient of the set of geodesic rays in $\mathbb{X}$ by the equivalence relation "remaining at bounded distance". The group G acts on $\partial_{\mathrm{vis}}\mathbb{X}$, and this action is transitive if and only if $\mathbb{X}$ has rank 1. Otherwise, each G-orbit identifies with a space of the form G/P where P is a parabolic subgroup, see [Kna96, Section VII.7] or [Bor91, Section IV.14].

1.3.2 Weyl chamber and Cartan projection

Choosing a connected component C of $\mathfrak{a} \setminus \bigcup_{\lambda\in\Lambda} \ker\lambda$, called a *Weyl chamber* of $\mathfrak{a}$, is equivalent to choosing a subset of positive roots $\Lambda^+ \subset \Lambda$, by declaring that Λ^+ is the set of roots taking positive values on C. We then have $\Lambda = \Lambda^+ \cup (-\Lambda^+)$. We fix such a connected component C and we denote by $\mathfrak{a}^+$ its closure.

Proposition 1.41 (Cartan decomposition, see [Kna96, Theorem VII.7.39]). *For every element $g \in G$, there exist two elements $k, k' \in K$ and a unique element $a \in \mathfrak{a}^+$ such that*

$$g = k\exp(a)k'.$$

This decomposition allows to define a map

$$\mu : G \to \mathfrak{a}^+$$

called the *Cartan projection* of G to $\mathfrak{a}^+$, which sends $g \in G$ to the element $a \in \mathfrak{a}^+$ given by the Cartan decomposition. Recall that $\mathfrak{a}$ is naturally a normed vector space, being a subspace of $\mathfrak{p} \simeq T_o\mathbb{X}$. The norm of the Cartan projection satisfies:

$$\forall g \in G, \qquad \|\mu(g)\| = d(o, g\cdot o). \tag{1.14}$$

The projectivization of $\mathfrak{a}^+$ has the structure of a complete simplicial complex. A *wall* is defined as a codimension 1 face of this complex, and a *vertex* is a face of dimension 0, corresponding to an extremal half-line of the cone $\mathfrak{a}^+$.

1.3.3 Complex structure

With the notation of the previous paragraph, the symmetric space $\mathbb{X}$ is called *Hermitian* if it admits a complex structure J for which G acts by biholomorphisms on $\mathbb{X}$. The following two results are classical.

Proposition 1.42 ([Kna96, Theorem 7.117]). *Let $\mathbb{X} = G/K$ be a symmetric space of noncompact type associated with a simple Lie group G. Then the center $\mathfrak{z}(\mathfrak{k})$ of the Lie algebra of K is of dimension 0 or 1, and it is of dimension 1 if and only if $\mathbb{X}$ is a Hermitian symmetric space. In this case, there exists an element $Z_0 \in \mathfrak{z}(\mathfrak{k})$ giving the complex structure on $\mathfrak{p}$ by*

$$J = ad(Z_0)|_{\mathfrak{p}}.$$

In particular J is uniquely determined up to a sign.

The following result is about G-invariant differential forms on symmetric spaces.

Proposition 1.43 ([Bur+05, Lemma 2.1]). *Let $\mathbb{X} = G/K$ be a symmetric space. Then every G-invariant differential form is closed.*

Let $\mathbb{X}$ be a Hermitian symmetric space of noncompact type, g its Riemannian metric and J its complex structure. Then the 2-form $\omega := g(\cdot, J\cdot)$ is positive and it follows from the last proposition that it is also closed. Hence $\mathbb{X}$ is a Kähler manifold. Since it is also a Cartan-Hadamard manifold, we deduce using Section 1.1.4 that it is a Stein manifold.

We assume from now on that $\mathbb{X}$ is Hermitian of noncompact type. Then there exists a biholomorphism, called the *Harish-Chandra embedding*, between $\mathbb{X}$ and a bounded open subset

Ω of an affine space $\mathbb{C}^N$, see [Kna96, Theorem 7.129]. This embedding extends to the visual boundary of $\mathbb{X}$. Now recall that the *Shilov boundary* of a bounded domain $D \subset \mathbb{C}^n$ is defined as the smallest closed subset $F \subset \partial D$ satisfying the maximum principle:

$$\forall f \in \mathcal{O}(D) \cap \mathcal{C}^0(\overline{D}), \quad \max_{\overline{D}} |f| = \max_F |f|.$$

The Shilov boundary of Ω is the image of a unique G-orbit in the visual boundary $\partial_{\mathrm{vis}}\mathbb{X}$, called the Shilov boundary of $\mathbb{X}$ and denoted by $\partial_{\mathrm{Shi}}\mathbb{X}$. The Shilov boundary can be written $\partial_{\mathrm{Shi}}\mathbb{X} = G/P$, where P is a maximal parabolic subgroup. In particular, in the projectivization of $\mathfrak{a}^+$ described in the previous subsection, every other G-orbit that is not contained in the wall opposite to the vertex corresponding to the Shilov boundary of $\mathbb{X}$ can be written $F = G/P'$ with $P' \subset P$. This defines a projection

$$\pi_F \colon F \to \partial_{\mathrm{Shi}}\mathbb{X}. \tag{1.15}$$

We will use the following structural results, for which we refer to [Kna96, Section VII.9], [RV73, Appendix] or [CØ03, Section 1]. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$, which is also a Cartan subalgebra of $\mathfrak{g}$. By maximality, $\mathfrak{h}$ contains Z_0. We denote by $\mathfrak{g}_{\mathbb{C}} =: \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-$ the decomposition of $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes \mathbb{C}$ into eigenspaces of $\mathrm{ad}(Z_0)_{\mathbb{C}}$ corresponding to the eigenvalues i, 0, and $-i$ respectively, where Z_0 is the element given by Proposition 1.42. A root λ of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{h}_{\mathbb{C}}$ is said to be *positive noncompact* if its eigenspace $\mathfrak{g}_{\mathbb{C}}^{\lambda}$ is contained in $\mathfrak{p}^+$. There then exists a family $(\gamma_1, \dots, \gamma_r)$ of positive noncompact roots relative to $\mathfrak{h}_{\mathbb{C}}$ and elements $H_s \in \mathfrak{h}$, $E_s \in \mathfrak{g}_{\mathbb{C}}^{\lambda_s}$ such that, setting

$$E_{-s} := \overline{E_s} \in \mathfrak{g}_{\mathbb{C}}^{-\lambda_s}, \quad X_s := E_s + E_{-s}, \quad Y_s := -i(E_s - E_{-s}),$$

the following properties hold (see [RV73, Appendix]):

1. $X_s, Y_s \in \mathfrak{p}$ and $Y_s = JX_s$.
2. $[H_s, X_s] = -2Y_s, \quad [H_s, Y_s] = 2X_s, \quad [X_s, Y_s] = 2H_s$.
3. For $s \neq t$, the subalgebras generated by (X_s, Y_s) and (X_t, Y_t) commute.
4. $\mathfrak{a} := \bigoplus_{s=1}^r \mathbb{R}X_s$ is a Cartan subalgebra contained in $\mathfrak{p}$.
5. Denoting by $(\xi_1, \dots, \xi_r)$ the basis of $\mathfrak{a}^*$ dual to $(X_1, \dots, X_r)$, the set of restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$ is $\Sigma := \pm\Sigma^+$ with either

   $$\Sigma^+ = \{\xi_p, 2\xi_p,\ \xi_s + \xi_t,\ \xi_s - \xi_t \mid 1 \le p \le r \text{ and } 1 \le s < t \le r\},$$

   in which case $\mathbb{X}$ is said to be of *non-tube type* and the root system of G is $(BC)_r$, or

   $$\Sigma^+ = \{2\xi_p,\ \xi_s + \xi_t,\ \xi_s - \xi_t \mid 1 \le p \le r \text{ and } 1 \le s < t \le r\},$$

   in which case $\mathbb{X}$ is said to be of *tube type* and the root system of G is C_r.
6. The roots $\pm 2\xi_s$ have multiplicity 1, the roots $\pm\xi_s \pm \xi_t$ have the same multiplicity $a \ge 1$, and the possible roots $\pm\xi_s$ have the same multiplicity $b \ge 0$, the values of a and b depending on the symmetric space.
7. The Shilov boundary of $\mathbb{X}$ is the G-orbit of the vector

   $$X = \sum_{s=1}^r X_s.$$

   We also define $Y = JX$.

Moreover, we recall that simple symmetric spaces have been classified by Cartan, and that there are:

- Four families and one exceptional simple Lie group whose associated symmetric space is Hermitian of tube type, namely the groups $\mathrm{SU}(n,n), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{SO}^*(4n), \mathrm{SO}(2,n)$ and $\mathrm{E}_7(-25)$.
- Two families and one exceptional simple Lie group whose associated symmetric space is Hermitian of nontube type, namely the groups $\mathrm{SU}(p,q)$ $(p<q), \mathrm{SO}^*(4n+2)$ and $\mathrm{E}_6(-14)$.

Definition 1.44. We call $\breve{\alpha} := 2\xi_r$ the *root associated to the Shilov boundary of* $\mathbb{X}$.

We will need the following result later, whose proof is included for the reader's convenience.

Lemma 1.45 ([RV73, Theorem 4.3])**.** *Let X be as defined above. For $k \in \{0,1,2\}$, set*

$$\mathfrak{p}^{(k)} := (\mathfrak{g}^{(k)} \oplus \mathfrak{g}^{(-k)}) \cap \mathfrak{p},$$

where $\mathfrak{g}^{(\pm k)}$ is the eigenspace of $\mathrm{ad}(X)$ associated to the eigenvalue $\pm k$. Then the complex structure J relates the spaces $\mathfrak{p}^{(0)}, \mathfrak{p}^{(1)}$ and $\mathfrak{p}^{(2)}$ in the following way:

$$J\mathfrak{p}^{(0)} = \mathfrak{p}^{(2)} \quad \textit{and} \quad J\mathfrak{p}^{(1)} = \mathfrak{p}^{(1)}.$$

Proof. For any pair $(k,l) \in \{0,1,2\}^2$, it is easy to verify that $[\mathfrak{g}^{(k)}, \mathfrak{g}^{(l)}] \subset \mathfrak{g}^{(k+l)}$, with the convention that $\mathfrak{g}^{(k+l)} = \{0\}$ if $k+l \notin \{0,1,2\}$. We define

$$\mathfrak{b} := \mathfrak{a} \oplus \bigoplus_{\lambda \in \Lambda^+} \mathfrak{g}^{\lambda} = \mathfrak{b}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$$

with

$$\mathfrak{b}^{(0)} := \mathfrak{a} \oplus \bigoplus_{s<t} \mathfrak{g}^{\xi_s - \xi_t}.$$

The Iwasawa decomposition gives $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$. Consequently, there exists a vector space isomorphism $\rho : \mathfrak{p} \to \mathfrak{b}$ that associates to $V \in \mathfrak{p}$ the unique element $\rho(V) \in \mathfrak{b}$ such that $V - \rho(V) \in \mathfrak{k}$. Moreover, $\rho(\mathfrak{p}^{(0)}) = \mathfrak{a}$, $\rho(\mathfrak{p}^{(1)}) = \mathfrak{g}^{(1)}$, and $\rho(\mathfrak{p}^{(2)}) = \mathfrak{g}^{(2)}$. We denote by $j := \rho J \rho^{-1}$ the complex structure induced on $\mathfrak{b}$. We will show that $j\mathfrak{b}^{(0)} = \mathfrak{g}^{(2)}$ and $j\mathfrak{g}^{(1)} = \mathfrak{g}^{(1)}$.

For every $s \in \{1,\dots,r\}$, $jX_s = \rho(JX_s) = \rho(Y_s)$. We claim that $\rho(Y_s) = H_s + Y_s$. Indeed, $Y_s = -H_s + (H_s + Y_s)$, so it suffices to show that $H_s + Y_s \in \mathfrak{b}$. Let $X' = \sum_{t=1}^r \alpha_t X_t$ with $\alpha_1, \dots, \alpha_r \in \mathbb{R}$. Then:

$$\begin{aligned}[X', H_s + Y_s] &= \sum_{t=1}^r \alpha_t [X_t, H_s + Y_s] = \alpha_s [X_s, H_s + Y_s] = \alpha_s (2Y_s + 2H_s) \\ &= 2\xi_s(X')(H_s + Y_s).\end{aligned}$$

Therefore, $H_s + Y_s \in \mathfrak{g}^{2\xi_s} \subset \mathfrak{b}^+$. Thus $jX = \sum_{s=1}^r (H_s + Y_s) \in \mathfrak{b}^{(2)}$.

The complex structure j is integrable, and hence satisfies the identity:

$$\forall V, W \in \mathfrak{b}, \quad [V,W] + j[jV,W] + j[V,jW] - [jV,jW] = 0.$$

Let $W \in \mathfrak{g}^{(2)}$. Then $[X, W] = 2W$. Moreover, $jX \in \mathfrak{g}^{(2)}$ so $[jX, W] = 0$. It follows that:

$$2W + j[X, jW] - [jX, jW] = 0.$$

Set $V = [X, jW]$. Then $V \in \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)}$ and $jV = [jX, jW] + 2W \in \mathfrak{g}^{(2)}$. We deduce that $[V, jV] = 0$, which implies $V = 0$. Therefore, $jW \in \mathfrak{b}^{(0)}$. Moreover, $\dim \mathfrak{b}^{(0)} = \dim \mathfrak{b}^{(2)} = r + \frac{ar(r-1)}{2}$, so by dimension count we have $j\mathfrak{b}^{(0)} = \mathfrak{g}^{(2)}$. Furthermore, for every $W_0 \in \mathfrak{b}^{(0)}$, we have $[jX, W_0] - 2jW_0 = 0$, hence $[jX, W_0] = 2jW_0$.

Let $W \in \mathfrak{g}^{(1)}$. Then $[X, W] = W$. Moreover, $jX \in \mathfrak{g}^{(2)}$ so $[jX, W] = 0$. We deduce:

$$W + j[X, jW] - [jX, jW] = 0,$$

and hence:

$$jW - [X, jW] - j[jX, jW] = 0.$$

We decompose $jW = Z_0 + Z_1 + Z_2$ with $Z_0 \in \mathfrak{b}^{(0)}, Z_1 \in \mathfrak{g}^{(1)}$, and $Z_2 \in \mathfrak{g}^{(2)}$. Then $[X, jW] = Z_1 + 2Z_2$ and $[jX, jW] = [jX, Z_0] = -2jZ_0$. We deduce:

$$Z_0 + Z_1 + Z_2 - (Z_1 + 2Z_2) - 2Z_0 = 0.$$

Hence, $Z_0 = 0$ and $Z_2 = 0$. Thus, $jW \in \mathfrak{g}^{(1)}$. □

1.3.4 Busemann functions in higher rank

Our reference for this section is [Dav23]. We fix a symmetric space of noncompact type $\mathbb{X}$, endowed with the metric induced by the Killing form. Let $\xi \in \partial_{\mathrm{vis}}\mathbb{X}$ and $y \in \mathbb{X}$. Then there is a unique geodesic ray γ in the class ξ such that $\gamma(0) = y$. For every $x \in \mathbb{X}$, the function $t \mapsto d_{\mathbb{X}}(x, \gamma(t)) - t$ is decreasing and bounded below, hence converges to a limit, denoted by $B_\xi(x, y) \in \mathbb{R}$.

Definition 1.46. The map $(x, y) \mapsto B_\xi(x, y)$ is called the *Busemann function* at ξ.

As in rank one, the Busemann functions satisfy the cocycle relation:

$$B_\xi(x, z) = B_\xi(x, y) + B_\xi(y, z),$$

for $x, y, z \in \mathbb{X}$, and in particular, the gradient and the Hessian of $B_\xi := B_\xi(\cdot, o)$ does not depend on the choice of $o \in \mathbb{X}$. We therefore speak of the gradient and the Hessian of B_ξ by a slight abuse of notation. The gradient of B_ξ is given by

$$\nabla B_\xi(x) = -v_{x,\xi},$$

where $v_{x,\xi}$ is the unit vector in $T_x\mathbb{X}$ pointing towards $\xi \in \partial_{\mathrm{vis}}\mathbb{X}$, see [Dav23, Proposition 4.5]. In particular, if $\mathbb{X}$ is Hermitian with Hermitian product denoted by $\langle \cdot, \cdot \rangle$, then for all $\xi \in \partial_{\mathrm{vis}}\mathbb{X}$ and for all $v \in T_x\mathbb{X}$, we have

$$dB_\xi(v)^2 + dB_\xi(Jv)^2 = |\langle v, v_{x,\xi} \rangle|^2 \leq \|v\|^2. \tag{1.16}$$

The Hessian of the Busemann function is given by the following result of Davalo.

Proposition 1.47 ([Dav23, Lemma 4.9])**.** *Let $w \in \mathfrak{a}$ be a unit vector and $z \in \mathfrak{p}$. Write z in the root decomposition of $\mathfrak{g}$ as*

$$z = z^0 + \sum_{\lambda \in \Lambda} z^\lambda.$$

Then the Hessian of the Busemann function B_{ξ_w} at ξ_w is computed by

$$D^2 B_{\xi_w}(z, z) = \sum_{\lambda \in \Lambda} |\lambda(w)| \|z^\lambda\|^2.$$

1.3.5 Critical exponent and Patterson–Sullivan measures in higher rank

The higher rank Patterson–Sullivan theory was first developed in the articles [Bur93; Alb99; Qui02a; Qui02b]. The next two subsections recall the results of this theory that will be needed in the sequel of this thesis.

Let Γ be a discrete and torsion-free subgroup of G. Its *critical exponent* can be defined as in rank one by the folllowing formula

$$\delta(\Gamma) = \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma o)} < \infty\},$$

where o is an arbitrary point in $\mathbb{X}$. It does not depend on the choice of o.

As in rank one, we define a *Patterson–Sullivan measure* for ρ as a $\rho(\Gamma)$-conformal density of dimension $\delta(\Gamma)$, that is, a family of measures $(\mu_x)_{x\in\mathbb{X}}$ on $\partial_{\mathrm{vis}}\mathbb{X}$ such that $\gamma_*\mu_x = \mu_{\gamma x}$ for all $x \in \mathbb{X}$ and $\gamma \in \Gamma$, and such that

$$\forall x, y \in \mathbb{X}, \quad \frac{d\mu_x}{d\mu_y} = e^{-\delta(\Gamma)B_\bullet(x,y)}, \tag{1.17}$$

where $B_\bullet(x, y)$ is the Busemann function defined in Subsection 1.3.4. As noticed by Albuquerque in [Alb99], the construction of Patterson–Sullivan measures done in Section 1.2.8 works in a higher rank setting. As a consequence, there always exist a Patterson–Sullivan measure for Γ.

Definition 1.48 (See [DK22, Definition **(RA)** in Section 1.6])**.** The group Γ is called *nonelementary* if the support of its Patterson–Sullivan measure contains at least three points.

1.3.6 Quint growth indicator function and entropy

Let Γ be a discrete and torsion-free subgroup of G. The *Quint growth indicator function* ψ_Γ, defined in [Qui02a] and [Qui02b], is a homogenous function on $\mathfrak{a}$ that can be seen as a generalization of the critical exponent in higher rank that contains more information than the critical exponent defined in the previous subsection (see Proposition 1.49 below for a precise link between the two notions). We now describe the construction of this function.

Recall that the Cartan projection $\mu : G \to \mathfrak{a}^+$ defined in Subsection 1.3.2 satisfies relation (1.14). Consequently, the series defining the critical exponent of a discrete subgroup of G can be rewritten

$$\sum_{\gamma\in\Gamma} e^{-s\|\mu(\gamma)\|}$$

For any open cone $\mathcal{C} \subset \mathfrak{a}$, we define $\tau_{\mathcal{C}}$ as

$$\tau_{\mathcal{C}} := \inf\{s \in \mathbb{R} \mid \sum_{\substack{\gamma\in\Gamma \\ \mu(\gamma)\in\mathcal{C}}} e^{-s\|\mu(\gamma)\|} < +\infty\} \in \mathbb{R} \cup \{-\infty\}.$$

Then, for any vector $v \in \mathfrak{a} \setminus \{0\}$, the value $\psi_\Gamma(v)$ is defined by

$$\psi_\Gamma(v) = \|v\| \inf_{\mathcal{C}} \tau_{\mathcal{C}} \in \mathbb{R} \cup \{-\infty\},$$

where the infimum is taken over all the open cones containing v.

Dually, the *entropy* associated with a linear form $\phi \in \mathfrak{a}^*$ was introduced in [Bur93; Alb99; Qui02a; Qui02b].The entropy $h_\Gamma(\lambda)$ of a linear form $\lambda \in \mathfrak{a}^*$ is defined by

$$h_\Gamma(\lambda) = \inf\{s \in \mathbb{R}_+ \mid \sum_{\gamma\in\Gamma} e^{-s\lambda(\mu(\gamma))} < \infty\}.$$

These two notions are linked through the following relation.

Proposition 1.49 ([Qui02a, Lemme III.1.3], see also [PSW23, Proposition 5.3])**.** *For every linear form $\lambda \in \mathfrak{a}$, we have*

$$h_\Gamma(\lambda) = \sup_{w\in\mathfrak{a}\setminus\{0\}} \frac{\psi_\Gamma(w)}{\lambda(w)}.$$

Moreover, this result holds more generally if one replaces the linear form λ by a 1-positively homogeneous function. When applied to the norm on $\mathfrak{a}$ induced by the Riemannian norm on $\mathbb{X}$, we obtain the following equality

$$\delta(\Gamma) = \sup_{w\in\mathfrak{a}\setminus\{0\}} \frac{\psi_\Gamma(w)}{\|w\|}.$$

This gives the first part of the following result.

Theorem 1.50 ([Qui02b, Proposition 8.10])**.** *There exists a unique unit vector $w_{\max} \in \mathfrak{a}^+$ such that*

$$\delta(\Gamma) = \psi_\Gamma(w_{\max}).$$

Moreover, Patterson–Sullivan measures constructed as in the previous subsection are supported on the G-orbit of the class of the geodesic ray $t \in \mathbb{R}_+ \mapsto \exp_o(tw_{\max})$.

Combining the two previous results, we also get the following inequality, that we will use in the following chapter.

Corollary 1.51. *For any linear form $\lambda \in \mathfrak{a}^*$ such that $\lambda(w_{\max}) \neq 0$, the following inequality holds:*

$$\delta(\Gamma) \leq h_\Gamma(\lambda)\lambda(w_{\max}).$$

Chapter 2

Holomorphic properties of Hermitian symmetric spaces quotients

This chapter is composed of two parts. The first one, based on the preprint [Sar24], presents results about holomorphic convexity and/or the absence of compact analytic subvarieties of positive dimension for ball quotients. The second part presents analogous results in higher rank, carried out in collaboration with Colin Davalo and not yet (pre)published.

In this chapter, we freely use the notions and results presented in the previous chapter.

2.1 Quotients of the ball

We begin by recalling the results stated in the introduction. The two first results are related with conjecture 1 of Dey and Kapovich: if Γ is a discrete and torsion-free subgroup of $\mathrm{PU}(n,1)$ with critical exponent less than 2, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ should be a Stein manifold.

Theorem A. *Let Γ be a geometrically finite and torsion-free subgroup of* $\mathrm{PU}(n,1)$.

(a) If Γ is Gromov-hyperbolic, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex.

(b) If Γ is a free group or if $\delta(\Gamma) < 2$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

(c) If Γ preserves a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

Theorem B. *Let Γ be a discrete and torsion-free parabolic subgroup of* $\mathrm{PU}(n,1)$.

(a) If $\delta(\Gamma) < 2$ or if Γ preserves a totally real and totally geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$, then Γ is virtually Abelian.

(b) If Γ is virtually Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.

We actually get a complete characterization of discrete parabolic subgroups of $\mathrm{PU}(n,1)$ such that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold, based upon the work of Miebach [Mie24]. It is the following result.

Theorem 2.1. *Let Γ be a discrete and torsion-free parabolic subgroup of $\mathrm{PU}(n,1)$, identified as explained in Subsection 1.2.3 with a subgroup of $\mathrm{U}(n-1) \ltimes N$, and denote by π, respectively Π, the projection of $\mathrm{U}(n-1) \ltimes N$ onto $\mathrm{U}(n-1)$, respectively $\mathrm{U}(n-1) \ltimes \mathbb{C}^{n-1}$. Let Γ_1 be a finite index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian, given by Lemma 1.27. Set*

$$V_1 := \bigcap_{\gamma \in \Gamma_1} \ker(\mathrm{Id} - \pi(\gamma)),$$

and let $p \colon \mathbb{C}^{n-1} \to V_1$ be the orthogonal projection onto V_1. Finally, define

$$W_1 := \mathrm{Span}(p(b(\gamma)) \mid \gamma \in \Gamma_1),$$

where $b(\gamma) = \Pi(\gamma) \cdot 0 \in \mathbb{C}^{n-1}$. Then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold if and only if $W_1 \cap iW_1 = \{0\}$.

Our next result is about the case of discrete subgroups with critical exponent equal to 2, and it implies Conjecture 2 of Dey and Kapovich.

Theorem C. *Let Γ be a discrete and torsion-free subgroup of $\mathrm{PU}(n,1)$ with $\delta(\Gamma) = 2$. Suppose that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety of positive dimension. Then Γ is a complex Fuchsian group.*

The last result that we would like to highlight requires a condition introduced by [CMW23], which the authors describe as "mysterious." In the following statement, the Patterson–Sullivan measure $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ is said to have *subexponential growth* if for all $\eta > 0$, there exists a constant $C_\eta > 0$ such that

$$\forall x \in \mathbb{H}^n_{\mathbb{C}},\ \mu_x(\partial \mathbb{H}^n_{\mathbb{C}}) \leq C_\eta e^{\eta d(x,o)},$$

for some basepoint $o \in \mathbb{H}^n_{\mathbb{C}}$, see [CMW23, §1.4].

Proposition 2.2. *Let Γ be a discrete and torsion-free subgroup of $\mathrm{PU}(n,1)$ with $\delta(\Gamma) < 2$. Assume that the Patterson–Sullivan measure $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ has subexponential growth. Then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.*

For instance, if $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ has positive injectivity radius and if the *Bowen–Margulis measure* ν associated with $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ is finite, then the total masses $\mu_x(\partial X)$ are uniformly bounded, see [CMW23, Theorem 1.15]. Bowen–Margulis measures can be defined from Patterson–Sullivan measures as follows. There is a bijection, called *Hopf coordinates* on the unit tangent bundle $T^1\mathbb{H}^n_{\mathbb{C}}$ of $\mathbb{H}^n_{\mathbb{C}}$, that sends any vector $v \in T^1_x\mathbb{H}^n_{\mathbb{C}}$ to the element $(v^-, v^+, t) \in (\partial\mathbb{H}^n_{\mathbb{C}} \times \partial\mathbb{H}^n_{\mathbb{C}} \setminus \Delta) \times \mathbb{R}$, with Δ the diagonal of $\partial\mathbb{H}^n_{\mathbb{C}} \times \partial\mathbb{H}^n_{\mathbb{C}}$, where $v^-, v^+ \in \partial\mathbb{H}^n_{\mathbb{C}}$ are the endpoints of the geodesic defined by v, and $t := B_{v^+}(o,x)$ is a parameter indicating where on this geodesic is located the basepoint x of v, defined as the value at x of a suitably normalized Busemann function centered at v^+. In these coordinates, the Bowem–Margulis measure ν associated with $(\mu_x)_{x \in \mathbb{H}^n_{\mathbb{C}}}$ is defined by

$$\nu := (\phi\mu_o \otimes \mu_o) \otimes dt,$$

where $\phi : (v^-, v^+) \mapsto \exp(B_{v^-}(o,p) + B_{v^+}(o,p))$ for any point p belonging to the geodesic between v^- and v^+, and dt is the Lebesgue measure on $\mathbb{R}$. We refer to [CMW23, Section 6.4.2] for more details and references about this measure.

Some results, including Proposition 2.2, can be stated in the more general framework where (X, ω) is a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ is a group acting freely and properly discontinuously by holomorphic isometries on X.

The rest of this section is organized as follows. In Subsection 2.1.1, we begin by stating and proving results that hold in the more general setting described above, including Proposition 2.2. We then turn to parabolic quotients and prove Theorems 2.1 and B in Subsection 2.1.2. Next, we prove Theorem A in Section 2.1.3. Finally, Section 2.1.4 is devoted to the proof of Theorem C.

2.1.1 Quotients of negatively curved Kähler-Hadamard manifolds

Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. In this setting, our first result is Proposition 1.12, proved in Section 1.1.3, that we now restate here.

Proposition 1.12. *Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and let Γ be a group acting freely and properly discontinuously by holomorphic isometries on X. Suppose that there exists a closed geodesically convex subset C of X that is Γ-invariant. Then the compact connected analytic subvarieties of X/Γ of positive dimension are contained in C/Γ. If furthermore the action of Γ on C is cocompact, then X/Γ is holomorphically convex.*

In this more general setting, one can also define the critical exponent $\delta(\Gamma)$ of Γ by the same formula as for discrete subgroups of $\mathrm{PU}(n,1)$, namely by

$$\delta(\Gamma) = \inf\{s \in \mathbb{R}^+ \mid \sum_{\gamma\in\Gamma} e^{-sd(o,\gamma\cdot o)} < +\infty\},$$

where o is an arbitrary point of X and d is the Riemannian distance associated with ω. This number does not depend on the choice of $o \in X$. We obtain the following result, generalizing [DK20, Theorem 15].

Proposition 2.3. *Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. If $\delta(\Gamma) < 2$, then X/Γ admits a strictly plurisubharmonic function and in particular does not contain any compact subvariety of positive dimension.*

If moreover X has pinched Ricci curvature, then holomorphic functions on X/Γ separate points and define local coordinates at all points of X/Γ.

From the two previous propositions, we deduce the following result which generalizes the main result of [DK20] to this larger setting.

Corollary 2.4. *Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. If $\delta(\Gamma) < 2$ and Γ acts cocompactly on a Γ-invariant geodesically convex subset of X, then X/Γ is a Stein manifold.*

The proof of Proposition 2.3 is an application of the following lemma, which is crucial, because Theorem C will be an analysis of the equality case in the inequality given by this lemma, and because it will be the starting point for our higher rank results stated in Section 2.2. We are going to use a Patterson–Sullivan measure on ∂X associated with Γ, as explained in the remark ending Section 1.2.8.

Lemma 2.5. *Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ be a nonelementary group acting freely and properly discontinuously by holomorphic isometries on X. Denote by δ the critical exponent of Γ, by $(\mu_x)_{x\in X}$ a Patterson–Sullivan measure associated with Γ and by $\|\mu_x\|$ the total mass of the measure μ_x for every $x \in X$. Then the Γ-invariant function on X defined by $f(x) := -\ln\|\mu_x\|$ satisfies*

$$i\partial\overline{\partial}f \geq \delta\left(1-\frac{\delta}{2}\right)\omega.$$

Proof. For every $x \in X$, let $\overline{\mu_x}$ be the normalized probability measure $\overline{\mu_x} = \frac{\mu_x}{\|\mu_x\|}$. Fixing a point $o \in X$, denote by $B_\theta := B_\theta(\cdot, o)$ the Busemann function at a point $\theta \in \partial X$ which vanishes at o. From dominated convergence together with the $\mathcal{C}^2$-regularity of Busemann functions and Formula (1.13), it follows that f is of class $\mathcal{C}^2$. We will now compute the Levi form of f by first computing its Hessian at a point x. First, the differential of f at a point $x \in X$ is given by

$$df(x) = \frac{\int_{\partial X} \delta dB_\theta(x) e^{-\delta B_\theta(x)} d\mu_o(\theta)}{\|\mu_x\|}.$$

Then, its Hessian is computed as follows.

$$\begin{aligned} D^2f(x) = \frac{1}{\|\mu_x\|}\int_{\partial X}\left(\delta D^2B_\theta(x)e^{-\delta B_\theta(x)} - \delta^2 dB_\theta(x)\otimes dB_\theta(x)e^{-\delta B_\theta(x)}\right)d\mu_o(\theta) + \\ \frac{\delta^2}{\|\mu_x\|^2}\left(\int_{\partial X} dB_\theta(x)e^{-\delta B_\theta(x)}d\mu_o(\theta)\right)^2 \\ = \delta\int_{\partial X} D^2B_\theta(x)d\overline{\mu_x}(\theta) + \delta^2\left(\left(\int_{\partial X} dB_\theta(x)d\overline{\mu_x}(\theta)\right)^2 - \int_{\partial X} dB_\theta(x)\otimes dB_\theta(x)d\overline{\mu_x}(\theta)\right). \end{aligned}$$

Let $v \in T_xX$. Using Identity (1.1), we obtain

$$i\partial\overline{\partial}f(v,Jv) \geq \delta\int_{\partial X} i\partial\overline{\partial}B_\theta(v,Jv)d\overline{\mu_x}(\theta) - \frac{\delta^2}{2}\int_{\partial X}(dB_\theta(v)^2 + dB_\theta(Jv)^2)d\overline{\mu_x}(\theta).$$

It is easily verified that

$$dB_\theta(v)^2 + dB_\theta(Jv)^2 \leq \|v\|^2,$$

where $\|v\|^2 := \omega(v,Jv)$, see Formula (1.16). Moreover, we have

$$i\partial\overline{\partial}B_\xi \geq \omega,$$

as a consequence of [GW79, Proposition 2.28] and Equality (1.9), see also [SY82; Che13].

Using these two inequalities, we deduce that

$$i\partial\overline{\partial}f \geq \delta(1-\frac{\delta}{2})\omega.$$

□

Proof of Proposition 2.3. If Γ is elementary, the result is already known, see [Che13, Theorem 1.1 and Proposition 1.3]. Assume that Γ is nonelementary and that $\delta < 2$. Then the function f defined in Lemma 2.5 is strictly plurisubharmonic. Suppose moreover that there is a constant $C > 0$ such that

$$\mathrm{Ricci}(\omega) \geq -C\omega(\cdot, J\cdot).$$

Then there is a constant $C' > 0$ such that

$$i\partial\overline{\partial}(C'f) + \mathrm{Ricci}(\omega)(\cdot, J\cdot) > 0.$$

Using [Che13, Proposition 4.1], we obtain that the holomorphic functions on X/Γ separate points and give local coordinate systems. □

Remark. A strictly plurisubharmonic function on X/Γ can also be constructed following Dey and Kapovich's ideas, providing an alternative proof for the first point of Proposition 2.3. Here is an outline of the argument. Let ϕ be the function defined on X by $\phi(x) := \tanh(d(o, x))^2$ for some basepoint $o \in X$. An application of the comparison result [GW79, Theorem A] together with Formula (1.1) gives that ϕ is strictly plurisubharmonic on X: this is obtained by comparing the Hessian of ϕ with the Hessian of the function $\widetilde{\phi}$ defined on $\mathbb{H}^{2n}_{\mathbb{R}}$, the real hyperbolic space of dimension $2n$, by $\widetilde{\phi}(x) := \tanh(d_{\mathrm{hyp}}(\widetilde{o}, x))^2$ for some basepoint $\widetilde{o}$ of $\mathbb{H}^{2n}_{\mathbb{R}}$. Then because

$$0 \leq 1 - \phi \leq 4e^{-2d(o,\cdot)},$$

we deduce that the convergence of the series

$$\sum_{\gamma \in \Gamma} e^{-2d(o,\gamma o)}$$

implies that

$$\sum_{\gamma \in \Gamma} (\phi(\gamma \cdot x) - 1)$$

converges uniformly on compact subsets of X to a strictly plurisubharmonic and Γ-invariant function ψ. When $X = \mathbb{H}^n_{\mathbb{C}}$, ϕ is the squared euclidean norm on the unit ball, and the above series is the one constructed in [DK20].

Remark. We already noticed that Corollary 2.4 follows directly from Propositions 1.12 and 2.3. Alternatively, each one of the three strictly plurisubharmonic functions $\phi + e^f$, $\phi + e^\psi$ and f is an exhaustion on X/Γ, with ϕ as in the proof of Proposition 1.12, f as in the proof of Proposition 2.3 and ψ as in the remark above. For the function f, this can be shown using that $(X \cup \Omega(\Gamma))/\Gamma$ is compact, where $\Omega(\Gamma) \subset \partial X$ is the domain of discontinuity of Γ.

The following proposition is a generalization of Proposition 2.2.

Proposition 2.6. *Let (X, ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ a nonelementary group acting freely and properly discontinuously by holomorphic isometries on X. Assume that $\delta(\Gamma) < 2$, X has pinched sectional curvature and the Patterson–Sullivan measure $(\mu_x)_{x \in X}$ has subexponential growth. Then X/Γ is a Stein manifold.*

Proof. Let f be the Γ-invariant function defined in Lemma 2.5 and o be a point of X/Γ. By [Tam10] or [Cho+10, §4 of Ch. 26], there is a constant $C > 0$ and a smooth function $g\colon X/\Gamma \to \mathbb{R}$ such that $g \geq d(\cdot, o)$ and $D^2(g) \geq -C\omega(\cdot, J\cdot)$. Using Identity (1.1), we deduce that $i\partial\overline{\partial}g \geq -C\omega$. Now define

$$\phi = g + C'f,$$

where C' is a positive constant such that

$$-C + C'\delta(1 - \frac{\delta}{2}) > 0.$$

Then ϕ is strictly plurisubharmonic by Lemma 2.5. Since $(\mu_x)_{x\in X}$ has subexponential growth, there is a constant $C''>0$ such that

$$f \geq -\frac{1}{2C'} d(\cdot, o) - \ln C''.$$

We obtain that

$$\phi \geq \frac{1}{2} d(\cdot, o) - \ln C'',$$

which implies that ϕ is an exhaustion. Thus X/Γ is a Stein manifold. □

We now summarize known criteria for asserting that X/Γ does not contain any compact subvariety of positive dimension.

Proposition 2.7. *Let (X,ω) be a complete simply connected Kähler manifold with sectional curvature bounded above by -1, and Γ a group acting freely and properly discontinuously by holomorphic isometries on X. Denote by d the Riemannian distance associated with ω. Then each one of the following condition is sufficient to assert that X/Γ does not contain any compact subvariety of positive dimension.*

(a) The group Γ is parabolic.

(b) The critical exponent of Γ satisfies $\delta(\Gamma)<2$.

(c) There exists a Γ-invariant geodesically convex subset C of X which is included in a totally real submanifold M of X.

(d) The Kähler form ω is exact on X/Γ.

(e) The cohomology group $H^2(\Gamma,\mathbb{R})$ vanishes.

(f) There is a complete vector field on X/Γ whose flow $(\phi_{-t})_t$ infinitesimally contracts complex subspaces in the sense that for every $x\in X/\Gamma$ and every nonzero linear complex subspace $V\subset T_x(X/\Gamma)$, we have

$$\left.\frac{d}{dt}\right|_{t=0} tr(d\phi_{-t}|_V) > 0.$$

Proof. (a) If Γ is parabolic, there is a Busemann function at some point $\xi\in\partial X$ which is invariant under the action of Γ, see [EO73, Proposition 7.8]. As a consequence X/Γ admits a strictly plurisubharmonic function.

(b) This follows from [DK20] for the complex hyperbolic space and from Proposition 2.3 in the general case.

(c) This is a consequence of Proposition 1.12. Indeed, let A be a compact connected subvariety of positive dimension included in X/Γ. Then $A\subset C/\Gamma$, and taking a smooth point x of the lift $\tilde{A}$ of A in X, we obtain that $T_x\tilde{A}\cap J(T_x\tilde{A})\subset T_xM$, which contradicts the hypothesis that M is totally real. Alternatively, it can be shown that in this case the distance squared function to the convex core is strictly plurisubharmonic, see [Che13, Theorem 1.1].

(d) This was shown in Lemma 1.7.

(e) The manifold X/Γ is a $K(\Gamma,1)$, and in particular its cohomology identifies with that of Γ, see [Bro82]. Therefore, if $H^2(\Gamma,\mathbb{R})=0$, the Kähler form ω is exact on X/Γ. This argument appears in a dual version in [MO18, Lemma 4.2], and also in [Kap22, page 27].

(f) See for instance [CMW23, Proof of Theorem 1.5]. □

Examples. 1. When $X = \mathbb{H}^n_{\mathbb{C}}$ and Γ is a discrete and torsion-free subgroup of $\mathrm{PO}(n,1)$, embedded in $\mathrm{PU}(n,1)$ so as to stabilize a copy of $\mathbb{H}^n_{\mathbb{R}}$, the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ does not contain any compact subvariety of positive dimension by the third point. In particular, as proven in [Che13], the quotient of X by the convex-cocompact group generated by a hyperbolic element is a Stein manifold.

2. Let $\Gamma = \pi_1(\Sigma_g)$ be a surface group and $\rho\colon \Gamma \to \mathrm{PU}(n,1)$ be a discrete and faithful representation of Γ in $\mathrm{PU}(n,1)$. Let τ be its Toledo invariant, defined in Section 1.2.5 Then ω is exact on $X/\rho(\Gamma)$ if and only if $\tau = 0$. This example appears in [Kap22, page 27].

3. If Γ is a free group, then $H^2(\Gamma,\mathbb{R}) = 0$ so X/Γ does not contain any compact subvariety of positive dimension.

4. Let Γ be a uniform and torsion-free lattice in $\mathrm{PU}(n,1)$. There is a holomorphic map A from $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ to an Abelian variety of dimension $N = \frac{b_1(\Gamma)}{2}$, called the Albanese variety of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, see [Voi02, chapter 12]. This map A lifts to a quasi-isometric holomorphic map $\widetilde{A}\colon \mathbb{H}^n_{\mathbb{C}}/[\Gamma,\Gamma] \to \mathbb{C}^N$, which implies that $\mathbb{H}^n_{\mathbb{C}}/[\Gamma,\Gamma]$ is holomorphically convex. This manifold is compact if and only if $b_1(\Gamma) = 0$. There are examples of uniform and torsion-free lattices Γ such that $\mathbb{H}^n_{\mathbb{C}}/[\Gamma,\Gamma]$ is not compact and contains compact subvarieties of positive dimension, see [CKY17]. However, for any uniform and torsion-free arithmetic lattice Γ in $\mathrm{PU}(n,1)$ with positive first Betti number, there is a finite index subgroup Γ_1 of Γ such that the Albanese map A_1 of $\mathbb{H}^n_{\mathbb{C}}/\Gamma_1$ is an immersion, see [LP24, Section 3.1] or [Eys18]. In particular, the lift $\widetilde{A_1}\colon \mathbb{H}^n_{\mathbb{C}}/[\Gamma_1,\Gamma_1] \to \mathbb{C}^{N_1}$ of A_1 is an immersion, where $N_1 = \frac{b_1(\Gamma_1)}{2}$, and we deduce that $\mathbb{H}^n_{\mathbb{C}}/[\Gamma_1,\Gamma_1]$ does not contain any compact subvariety of positive dimension. Thus $\mathbb{H}^n_{\mathbb{C}}/[\Gamma_1,\Gamma_1]$ is a Stein manifold.

2.1.2 Discrete parabolic subgroups of $\mathrm{PU}(n,1)$

2.1.2.1 A characterization of Stein parabolic quotients of the ball

In this subsection, we restate and prove Theorem 2.1.

Theorem 2.1. *Let Γ be a discrete and torsion-free parabolic subgroup of $\mathrm{PU}(n,1)$, identified as explained in Subsection 1.2.3 with a subgroup of $\mathrm{U}(n-1) \ltimes N$, and denote by π, respectively Π, the projection of $\mathrm{U}(n-1) \ltimes N$ onto $\mathrm{U}(n-1)$, respectively $\mathrm{U}(n-1) \ltimes \mathbb{C}^{n-1}$. Let Γ_1 be a finite index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian, given by Lemma 1.27. Set*

$$V_1 := \bigcap_{\gamma\in\Gamma_1} \ker(\mathrm{Id} - \pi(\gamma)),$$

and let $p\colon \mathbb{C}^{n-1} \to V_1$ be the orthogonal projection onto V_1. Finally, define

$$W_1 := \mathrm{Span}(p(b(\gamma)) \mid \gamma \in \Gamma_1),$$

where $b(\gamma) = \Pi(\gamma)\cdot 0 \in \mathbb{C}^{n-1}$. Then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold if and only if $W_1 \cap iW_1 = \{0\}$.

Before passing to the proof of this theorem, we want to highlight that in complex dimension 2, this characterization takes the following simpler form: a parabolic quotient $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ is Stein if and only if Γ is virtually Abelian (Corollary 2.9).

Proof of Theorem 2.1. Since $\mathbb{H}^n_{\mathbb{C}}/\Gamma_1$ is a finite covering of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, one of these two manifolds is Stein if and only if the other one is. To simplify the notation, we can thus assume without loss of generality that $\Gamma_1 = \Gamma$.

Step 1. Existence of $\Lambda \in \mathbb{C}^{n-1}$ such that $\forall \gamma \in \Gamma, (\Pi(\gamma) - \mathrm{Id})\Lambda \in V_1$.

The group $\pi(\Gamma)$ is Abelian, hence there is an orthonormal basis $e = (e_1, \dots, e_{n-1})$ of $\mathbb{C}^{n-1}$ as well as morphisms $a_1, \dots, a_{n-1}$ from Γ to the unit circle in $\mathbb{C}$ such that, in the basis e

$$\pi(\gamma) = \mathrm{Diag}(a_1(\gamma), \dots, a_{n-1}(\gamma)).$$

For $\gamma \in \Gamma$, let $(b_1(\gamma), \dots, b_{n-1}(\gamma))$ be the coordinates of $b(\gamma)$ in the basis e. Up to permuting the elements of e, we can assume that $(e_{k+1}, \dots, e_{n-1})$ forms a basis of V_1 for some integer $k \in \{0, \dots, n-1\}$. For all $i \in \{1, \dots, k\}$, there is an element $\gamma_i \in \Gamma$ such that $a_i(\gamma_i) \neq 1$. Set

$$\lambda_i := \frac{b_i(\gamma_i)}{1 - a_i(\gamma_i)}.$$

As $\Pi(\Gamma)$ is Abelian, we have

$$\forall \gamma \in \Gamma, \forall i \leq k, b_i(\gamma) = \lambda_i(1 - a_i(\gamma)).$$

We set $\Lambda := {}^t(\lambda_1, \dots, \lambda_k, 0, \dots, 0) \in \mathbb{C}^{n-1}$, so that

$$\forall \gamma \in \Gamma, (\Pi(\gamma) - \mathrm{Id})\Lambda = {}^t(0, \dots, 0, b_{k+1}(\gamma), \dots, b_{n-1}(\gamma)) \in V_1.$$

Step 2. Conjugation and separation of the elliptic and unipotent parts of Γ.

Let T_Λ be the element $(\mathrm{Id}, \Lambda, 0)$ of $\mathrm{U}(n-1) \ltimes N$ and, for all $\gamma \in \Gamma$, define $\phi(\gamma) := T_\Lambda^{-1}\gamma T_\Lambda$. A computation shows that for all $\gamma \in \Gamma$, we have

$$\phi(\gamma) = (\pi(\gamma), b(\phi(\gamma)), c(\phi(\gamma))), \text{ where } b(\phi(\gamma)) = p(b(\gamma)) \in V_1 \text{ and } c(\phi(\gamma)) \in \mathbb{R}.$$

Set

$$\phi(\gamma)_e := (\pi(\gamma), 0, 0) \text{ and } \phi(\gamma)_u := (\mathrm{Id}, b(\phi(\gamma)), c(\phi(\gamma))).$$

Using that $b(\phi(\gamma)) \in V_1$ for all $\gamma \in \Gamma$, we get

$$\forall \gamma, \gamma' \in \Gamma, [\phi(\gamma)_e, \phi(\gamma')_u] = \mathrm{Id}\,.$$

It is also easily verified that $\phi(\gamma) = \phi(\gamma)_e \phi(\gamma)_u$ for all $\gamma \in \Gamma$. More generally, $U(k)$, seen as a subgroup of $\mathrm{U}(n-1)$ fixing V_1 pointwise, commutes with $\phi(\gamma)_u$ for $\gamma \in \Gamma$. In particular, $\phi(\Gamma)_E := \{\phi(\gamma)_e \mid \gamma \in \Gamma\}$ and $\phi(\Gamma)_U := \{\phi(\gamma)_u \mid \gamma \in \Gamma\}$ are groups and $\phi(\Gamma)_U < N$. Moreover, $\phi(\Gamma)_U$ is discrete in N. Indeed, let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence in Γ such that $\phi(\gamma_k)_u \underset{k\to+\infty}{\longrightarrow} \mathrm{Id}$. After passing to a subgroup, we can assume that the sequence $\phi(\gamma_k)_e$ converges to a limit $M \in \mathrm{U}(n-1)$. Thus $\phi(\gamma_k)$ converges to M, and since $\phi(\Gamma)$ is discrete, this sequence has to be stationary. Hence $\phi(\Gamma)_U$ is discrete in N.

Step 3. Characterization of Stein quotients.

We can rewrite W_1 as

$$W_1 = \mathrm{Span}_{\mathbb{R}}(\{b(\phi(\gamma)) \mid \gamma \in \Gamma\}).$$

From [Mie24, Theorem 1.4], we obtain that the quotient $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is Stein if and only if W_1 is totally real.

Assume that W_1 is totally real. Then $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is a Stein manifold, so it has a strictly plurisubharmonic exhaustion function $\psi_U \colon \mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U \to \mathbb{R}_+$. Moreover, the holomorphic action of $U(k)$ on $\mathbb{H}^n_{\mathbb{C}}$ descends to the quotient $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$, and by averaging ψ_U over the orbits of $U(k)$, we can assume that ψ_U is $U(k)$-invariant. Then ψ_U lifts to a strictly plurisubharmonic function $\widetilde{\psi_U}\colon \mathbb{H}^n_{\mathbb{C}} \to \mathbb{R}_+$ which is invariant by $\phi(\Gamma)_U$ and $U(k)$. This function descends to a strictly plurisubharmonic function $\psi\colon \mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma) \to \mathbb{R}_+$, which is an exhaustion. Thus, $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$, and therefore $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, are Stein.

Conversely, if $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, hence $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$, is a Stein manifold, then $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)$ admits a strictly plurisubharmonic exhaustion function. This implies that $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ has a strictly plurisubharmonic exhaustion function, thus $\mathbb{H}^n_{\mathbb{C}}/\phi(\Gamma)_U$ is a Stein manifold. Therefore, W_1 is totally real. □

Using Theorem 2.1, we obtain as a corollary the second point of Theorem B.

Corollary 2.8. *Let Γ be a discrete and torsion-free parabolic subgroup of* $\mathrm{PU}(n,1)$*. If Γ is virtually Abelian, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.*

Proof. As in the proof of Theorem 2.1, we assume without loss of generality that Γ is Abelian and we decompose $\phi(\Gamma)$ into an elliptic part $\phi(\Gamma)_E$ and a unipotent part $\phi(\Gamma)_U$. Then $\phi(\Gamma)_U$ is a discrete and Abelian parabolic subgroup of $\mathrm{PU}(n,1)$. It is known that the quotient of the complex hyperbolic space by such a subgroup is a Stein manifold, see [Che13]. This is also a particular case of [Mie24, Theorem 1.4], because with the notation of Step 1 above, for all $\gamma, \gamma' \in \Gamma$, Formula (1.6) together with the fact that $b(\phi(\gamma))$ and $b(\phi(\gamma'))$ belong to V_1 imply that

$$0 = c(\phi(\gamma)\phi(\gamma')) - c(\phi(\gamma')\phi(\gamma)) = 2\Im\langle b(\phi(\gamma)), b(\phi(\gamma'))\rangle.$$

and this implies that W_1 is totally real, so that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold. □

2.1.2.2 Proof of Theorem B and parabolic subgroups of $\mathrm{PU}(2,1)$

Proof of Theorem B. The second point of the theorem is given by Corollary 2.8. For the first point, let first Γ be a discrete and torsion-free parabolic subgroup of $\mathrm{PU}(n,1)$ which is not virtually Abelian. We will show that $\delta(\Gamma) \geq 2$ by finding two elements $x, y \in \Gamma$ that generate a group of critical exponent equal to 2. Let us fix, as in Subsection 1.2.3, a basis f of $\mathbb{C}^{n+1}$ which induces an identification between Γ and a subgroup of $\mathrm{U}(n-1) \ltimes N$. Let Γ_1 be a finite index subgroup of Γ such that $\Pi(\Gamma_1)$ is Abelian, given by Lemma 1.27. Since the set of commutators of elements of Γ_1 is included in the kernel of Π, which coincides with the center $Z(N)$ of N, and Γ_1 is not Abelian, we deduce that Γ contains two elements x and y such that $\Pi(x)$ and $\Pi(y)$ commute, but x and y do not. Then according to Formula (1.12), the critical exponent of the group generated by x and y is $\frac{2l+k}{2}$, where $l \in \{0,1\}$ is the dimension of the $\mathbb{R}$-span of the elements $c(\gamma)$ for $\gamma \in \Gamma$ and $k \in \{0,1,2\}$ is the dimension of the $\mathbb{R}$-span of (x) and $b(y)$. Since x and y do not commute, we see that $l = 1$ and $k = 2$. Thus $\delta(\langle x, y\rangle) = 2$.

Assume now that Γ preserves a totally real geodesic submanifold of $\mathbb{H}^n_{\mathbb{C}}$. Then we can realize Γ as a discrete and virtually nilpotent subgroup of

$$\mathrm{P}((O(k-1) \ltimes \mathbb{R}^{k-1}) \times U(n-k))$$

for some integer $k \in \{1, \dots, n\}$. Consequently, Γ is virtually Abelian. □

Corollary 2.9. *Let Γ be a discrete and torsion-free parabolic subgroup of* $\mathrm{PU}(2,1)$*. Then $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ is Stein if and only if Γ is virtually Abelian.*

Proof. If Γ is not virtually Abelian, choose two elements $x, y \in \Gamma$ as in the proof of Theorem B. Since x and y do not commute, we get that $\Im\langle b(x), b(y)\rangle \neq 0$, and thus $W := \mathrm{Span}_{\mathbb{R}}(b(x), b(y)) \subset$

$\mathbb{C}$ is equal to $\mathbb{C}$. In particular W is not totally real and using Theorem 2.1, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a covering $\mathbb{H}^n_{\mathbb{C}}/\langle x_0, y_0\rangle$ which is not Stein. As any covering of a Stein manifold is Stein, see [Ste56; Siu78], this implies that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not Stein. □

2.1.2.3 Examples of parabolic quotients of the ball

In the following two examples, we fix, as in Subsection 1.2.3, a basis $f = (f_1, f_2, e_1, \ldots, e_{n-1})$ of $\mathbb{C}^{n+1}$ which induces an identification between parabolic subgroups of $\mathrm{PU}(n,1)$ fixing $[f_1]$ and subgroups of $\mathrm{U}(n-1) \ltimes N$.

Example. Here is an example of a discrete unipotent subgroup Γ of $\mathrm{PU}(n,1)$ with $\delta(\Gamma) = 2$, for which $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not holomorphically convex. The group Γ generated by $\gamma_1 := (\mathrm{Id}, e_1, 0)$ and $\gamma_2 := (\mathrm{Id}, ie_1, 0)$ is the set of all elements of the form

$$(\mathrm{Id}, (k_1 + ik_2)e_1, 2\ell - k_1k_2),$$

where $(k_1, k_2, \ell) \in \mathbb{Z}^3$. In particular, Γ is discrete, and Formula (1.12) shows that $\delta(\Gamma) = 2$. Finally, the quotient $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ naturally identifies with a bundle of punctured disks over the base $B := \mathbb{C}/(\mathbb{Z}+i\mathbb{Z}) \times \mathbb{C}^{n-2}$. If $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ were holomorphically convex, it would be Stein by Proposition 2.7-(a) and we would deduce that B is a Stein manifold by [CD97, Lemma 1.6], which is not the case. Therefore, $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not holomorphically convex.

Example. Here is an example of a complex hyperbolic bundle of punctured disks over a Cousin group. We work in dimension $n = 3$, but this example generalizes to any dimension $n \geq 3$. Using the identification introduced before the previous example, define three vectors in $\mathbb{C}^2 = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ by $v_1 = e_1$, $v_2 = e_2$, and $v_3 = ae_1 + be_2$ for $(a, b) \in \mathbb{C}^2$ two complex numbers such that

$$\begin{cases} \lambda := \Im(a) = \Im(b) \neq 0, \\ \Re(a) - \Re(b) \notin \mathbb{Q}. \end{cases}$$

The fact that $\Im(a) \neq 0$ and $\Im(b) \neq 0$ implies that v_1, v_2 and v_3 are $\mathbb{R}$-linearly independent, and both conditions together imply that $1, a$ and b are $\mathbb{Z}$-linearly independent. We deduce that the subgroup Γ_0 of $\mathbb{C}^2$ generated by v_1, v_2 and v_3 is discrete, and that the quotient $\mathbb{C}^2/\Gamma_0$ has no compact factor and does not admit any non-constant holomorphic function (see for example [Nap90, pages 451-452]). Let Γ be the subgroup of $\mathrm{U}(n-1) \ltimes N$ generated by the three elements $\gamma_i = (\mathrm{Id}, v_i, 0)$ for $i = 1, 2$ and 3. The equality $\Im(a) = \Im(b) = \lambda$ implies that

$$[\gamma_3, \gamma_1] = [\gamma_3, \gamma_2] = (\mathrm{Id}, 0, 2\lambda).$$

Any element of Γ is of the form $\gamma_1^{k_2}\gamma_2^{k_2}\gamma_3^{k_3}[\gamma_3, \gamma_1]^{\ell}$, with $(k_1, k_2, k_3, \ell) \in \mathbb{Z}^4$, and we deduce that Γ is the set of all elements of the form

$$(\mathrm{Id}, k_1v_2 + k_2v_2 + k_3v_3, ((k_1 + k_2)k_3 + 2\ell)\lambda).$$

with $(k_1, k_2, k_3, \ell) \in \mathbb{Z}^4$. Consequently, Γ is discrete, and $\mathbb{H}^3_{\mathbb{C}}/\Gamma$ is biholomorphic to a bundle of punctured disks over $\mathbb{C}^2/\Gamma_0$. Since $\mathbb{C}^2/\Gamma_0$ is not Stein, we deduce as in the previous example that $\mathbb{H}^3_{\mathbb{C}}/\Gamma$ is not holomorphically convex. Additionally, Formula (1.12) shows that $\delta(\Gamma) = \frac{5}{2}$.

2.1.3 Geometrically finite quotients of the ball

Throughout this section, we will freely use the decomposition (1.11) and the notation associated with this decomposition.

2.1.3.1 A characterisation of holomorphically convex geometrically finite quotients of the ball

An important step in the proof of Theorem A is given by the following result.

Proposition 2.10. *Let Γ be a geometrically finite and torsion-free subgroup of* $\mathrm{PU}(n,1)$*. The following are equivalent:*

1. *The manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a plurisubharmonic exhaustion function.*
2. *For any parabolic subgroup $P < \Gamma$, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is holomorphically convex, or equivalently a Stein manifold.*
3. *The manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex.*

For the proof of Proposition 2.10, we will need the following lemma about parabolic quotients of the ball.

Lemma 2.11. *Let P be a discrete and torsion-free parabolic subgroup of* $\mathrm{PU}(n,1)$*, and $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ a point fixed by P. The following statements are equivalent:*

1. *$\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold.*
2. *For any horoball $H \subset \mathbb{H}^n_{\mathbb{C}}$ centered at ξ, the quotient H/P is a Stein manifold.*
3. *There exists a horoball $H \subset \mathbb{H}^n_{\mathbb{C}}$ centered at ξ for which H/P is a Stein manifold.*

Proof. ($1 \implies 2$) If $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold, it admits a strictly plurisubharmonic exhaustion function $\psi\colon \mathbb{H}^n_{\mathbb{C}}/P \to \mathbb{R}_+$. The Busemann function $b\colon \mathbb{H}^n_{\mathbb{C}} \to \mathbb{R}$ at ξ is invariant under the action of P. Let $H_\lambda := b^{-1}((-\infty,\lambda))$ be a horoball centered at ξ. The function $\psi + \frac{1}{\lambda - b}$ defined on H_λ/P is a strictly plurisubharmonic exhaustion function on H_λ/P, which shows that H_λ/P is a Stein manifold.

The implication $2 \implies 3$ is immediate. We now show that $3 \implies 2$. We fix, as in Subsection 1.2.3, a basis $f = (f_1, f_2, e_3, \ldots, e_{n+1})$ of $\mathbb{C}^{n+1}$ which induces an identification between parabolic elements of $\mathrm{PU}(n,1)$ fixing $\xi = [f_1]$ and elements of $\mathrm{U}(n-1) \ltimes N$. The elements of P, seen as biholomorphisms of $\mathbb{CP}^n$, commute with the biholomorphisms $L_t\colon \mathbb{CP}^n \to \mathbb{CP}^n$ defined for all real numbers t in the basis f by the matrices

$$\begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{I}_{n-1} \end{pmatrix}.$$

With the notation of Subsection 1.2.3, it is easily checked that for any pair (λ,μ) of real numbers, the map L_t with $t = e^{-2\lambda} - e^{-2\mu}$ sends the horoball $H_\lambda := b^{-1}((-\infty,\lambda))$ to the horoball $H_\mu := b^{-1}((-\infty,\mu))$. We deduce that the quotients of horoballs centered at ξ by P are all biholomorphic.

($2 \implies 1$) The manifold $\mathbb{H}^n_{\mathbb{C}}/P$ is the union of a one-parameter family of Stein manifolds, so it is Stein by Theorem 1.16. □

We now come to the proof of Proposition 2.10.

Proof of Proposition 2.10. ($1 \implies 2$) Suppose that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a plurisubharmonic exhaustion function $\phi\colon \mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{R}$. Let P be a maximal parabolic subgroup of Γ. There exists a Busemann function B, invariant under P, such that the set $C := B^{-1}((-\infty,0))/P$ is biholomorphic to an open subset of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. The function $\phi|_C + \frac{-1}{B}$ is a strictly plurisubharmonic exhaustion function of C, and therefore C is a Stein manifold. Using Lemma 2.11, we deduce that $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. If now P is any parabolic subgroup of Γ, it is contained in a maximal parabolic subgroup P_0 of Γ. The manifold $\mathbb{H}^n_{\mathbb{C}}/P$ is a covering of $\mathbb{H}^n_{\mathbb{C}}/P_0$, which is Stein, and therefore $\mathbb{H}^n_{\mathbb{C}}/P$ is Stein.

($2 \implies 3$) This proof is inspired by [Che13, Proof of Theorem 1.4]. Suppose that, for every maximal parabolic subgroup $P < \Gamma$, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. Recall from (1.11) the decomposition

$$X_\Gamma =: Q \cup \bigcup_{i=1}^{k} E_i,$$

and the notation associated with this decomposition. We define $E_i' \subset E_i$ as the quotient of the horoball $B_i^{-1}((-\infty, -1))$ by P_i. Moreover, the convex core C_Γ of X_Γ has compact intersection with Q. By the arguments given in the proofs of Proposition 2.3 and Lemma 1.13, we get that the squared distance function to the convex hull $C(\Gamma)$ of the limit set descends to a convex function ϕ on X_Γ, which is strictly convex outside C_Γ. By Richberg's theorem, there exists a continuous plurisubharmonic function $\widetilde{\phi}$ which is smooth and strictly plurisubharmonic outside C_Γ, and such that

$$\phi \leq \widetilde{\phi} \leq \phi + \frac{1}{2},$$

see [Dem, Theorem I.5.21]. Moreover, Lemma 2.11 implies that for any $i \in \{1, \dots, k\}$, the open subset E_i of X_Γ is a Stein manifold, and admits a strictly plurisubharmonic exhaustion function. Let Ψ_i be a smooth non-negative function that coincides with this function on E_i' and vanishes outside E_i. For any integer $j \in \mathbb{N}$, let T_j^i be the compact subset of X_Γ defined by

$$T_j^i := \{x \in \overline{E_i} \setminus E_i' \mid j \leq \widetilde{\phi}(x) \leq j+1\}.$$

Then, as soon as $j \geq 1$, the function $\widetilde{\phi}$ is strictly plurisubharmonic on T_i^j, so there exists a constant $\beta_j^i > 0$ such that $i\partial\bar{\partial}\Psi_i \geq -\beta_j^i i\partial\bar{\partial}\widetilde{\phi}$ on T_j^i. It follows that there exists a strictly increasing convex function $\lambda\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lambda(t) \underset{t\to+\infty}{\longrightarrow} +\infty$ and such that

$$N := \lambda \circ \widetilde{\phi} + \sum_{i=1}^{k} \Psi_i$$

is strictly plurisubharmonic on the set

$$\bigcup_{i=1}^{k} \bigcup_{j \geq 1} T_j^i.$$

On Q, this function N coincides with $\lambda \circ \widetilde{\phi}$ and it is strictly plurisubharmonic on $Q \cap X_\Gamma \setminus C_\Gamma$. On each E_i', since Ψ_i is strictly plurisubharmonic and $\widetilde{\phi}$ is plurisubharmonic, N is strictly plurisubharmonic. In conclusion, N is strictly plurisubharmonic outside the compact set

$$(C_\Gamma \cap Q) \cup \bigcup_{i=1}^{k} T_0^i.$$

Moreover, N is an exhaustion function. Indeed, if $(x_n)_{n\in\mathbb{N}}$ is a sequence in X_Γ without accumulation point, then, after extracting a subsequence, it converges to the boundary ∂X_Γ or has values in one of the open sets E_i'. In the first case where $x_n \longrightarrow x_\infty \in \Omega(\Gamma)/\Gamma$, we claim that

$\phi(x_n) \longrightarrow +\infty$. Assuming the contrary, we obtain a sequence $(\widetilde{x_n})_{n\in\mathbb{N}}$ in $\mathbb{H}^n_{\mathbb{C}}$ converging to an element $\widetilde{x_\infty} \in \Omega(\Gamma)$, which remains at bounded distance from $C(\Gamma)$, and thus another sequence $(c_n)_{n\in\mathbb{N}}$ in $C(\Gamma)$ converging to $\widetilde{x_\infty}$. Thus, $\widetilde{x_\infty} \in \Omega(\Gamma) \cap \partial C(\Gamma)$. This is a contradiction, because $\partial C(\Gamma) = \Lambda(\Gamma)$, see [And83]. In the case where the sequence lies in E_i', it does not accumulate and therefore, after after passing to a subsequence, we have $\Psi(x_n) \to +\infty$. Thus N is an exhaustion function. Finally, a second application of Richberg's theorem shows that there exists a continuous exhaustion function $\widetilde{N}: X_\Gamma \to \mathbb{R}$ which is smooth and strictly plurisubharmonic outside a compact set. Therefore, X is holomorphically convex.

The implication $3 \implies 1$ is classical, see for example [Dem, Theorem I.6.14]. □

Remark. If we replace $\mathbb{H}^n_{\mathbb{C}}$ by a complete simply connected Kähler manifold (X, ω) with negatively pinched sectional curvature, and assume that Γ is a group acting freely and geometrically finitely by holomorphic isometries on X, I do not know if Lemma 2.11 remains true (the proof uses the holomorphic maps L_t whose existence is specific to the complex hyperbolic case). In Proposition 2.10, it remains true that $1 \iff 3$. To show that $1 \implies 3$, one argues as in the proof above, noticing that if X/Γ admits a plurisubharmonic exhaustion ϕ, then the open sets E_i appearing in the decomposition

$$X/\Gamma = Q \cup \bigcup_{i=1}^{k} E_i$$

are Stein manifolds, with a strictly plurisubharmonic exhaustion given by $\phi + \frac{-1}{B_i}$, where B_i is a Busemann function on X associated to a parabolic point corresponding to the cusp E_i.

2.1.3.2 Proof of Theorem A

For the proof of Theorem A-(a), we will need the following lemma, which is presumably classical, and the proof of which we include for completeness.

Lemma 2.12. *Let X be a complete simply connected Riemannian manifold with negatively pinched curvature, and P a discrete and torsion-free parabolic subgroup of isometries of X. Then P is cyclic or contains a copy of $\mathbb{Z}^2$.*

Proof of Lemma 2.12. By the Margulis lemma, P contains a finite index nilpotent subgroup P'. Moreover, P' is finitely generated according to [Bow93]. We distinguish two cases:

- If P' is Abelian, then P' is cyclic or contains a copy of $\mathbb{Z}^2$. Since a virtually cyclic torsion-free group is cyclic, we deduce that P is cyclic or contains a copy of $\mathbb{Z}^2$.

- Otherwise, let g be a non-trivial element in the center of P', and h an element of P' which does not belong to the center of P'. Then g and h generate a subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$. Suppose, by contradiction, that this group is cyclic. Then g and h are powers of an element $k \in P'$. In a torsion-free and finitely generated nilpotent group, the centralizers of an element and its powers coincide, and consequently g and h have the same centralizer in P'. This yields a contradiction, and consequently P contains a copy of $\mathbb{Z}^2$. □

Proof of Theorem A. Let Γ be a geometrically finite and torsion-free subgroup of $\mathrm{PU}(n,1)$. Assume first that Γ is Gromov-hyperbolic. Then Γ does not contain a copy of $\mathbb{Z}^2$, see [BH99, Corollary III.Γ.3.10], so according to Lemma 2.12, the non-trivial parabolic subgroups of Γ are cyclic. The quotient of the complex hyperbolic space by the action of a cyclic parabolic group is a Stein manifold, as follows, for example, from Theorem B, see also [dFI01] or [Mie10]. Proposition 2.10 implies that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex.

In particular, if Γ is free, then $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. Using Proposition 2.7-(e), we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

Suppose now that $\delta(\Gamma) < 2$. For any parabolic subgroup $P < \Gamma$, we have $\delta(P) \leq \delta(\Gamma) < 2$, so $\mathbb{H}^n_{\mathbb{C}}/P$ is Stein according to Theorem B. Using Proposition 2.10, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. Since $\delta(\Gamma) < 2$, this manifold does not contain any compact analytic subvariety of positive dimension according to [DK20, Theorem 15] or Proposition 2.3. We deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein.

Finally, suppose that Γ preserves a totally real and totally geodesic submanifold $\mathbb{H}^k_{\mathbb{R}} \subset \mathbb{H}^n_{\mathbb{C}}$. Then according to Theorem B, for any parabolic subgroup P of Γ, the quotient $\mathbb{H}^n_{\mathbb{C}}/P$ is a Stein manifold. Using Proposition 2.10, we deduce that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is holomorphically convex. This manifold does not contain any compact analytic subvariety of positive dimension, according for example to Proposition 2.7-(c). Thus $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is Stein. □

We conclude this section by a remark which follows from Lemma 2.12.

Remark. Let X be a complete simply connected Riemannian manifold with negatively pinched curvature, and Γ a group containing no copy of $\mathbb{Z}^2$. If Γ acts faithfully, discretely, and geometrically finitely by isometries on X, then Γ is Gromov-hyperbolic. Indeed, a geometrically finite group is hyperbolic relative to its parabolic subgroups. Under the assumption that Γ contains no copies of $\mathbb{Z}^2$, the parabolic subgroups of Γ are cyclic, and in particular Gromov-hyperbolic. This implies that Γ itself is Gromov-hyperbolic, see [Osi06].

We can use this fact to exhibit finitely generated groups which admit a discrete and faithful representation in $\mathrm{PU}(n,1)$ but no discrete, faithful, and geometrically finite representation of N in $\mathrm{PU}(n,1)$. To do this, let first Γ_0 be a cocompact arithmetic lattice of the simplest type of $\mathrm{PU}(n,1)$, for the definition of which we refer to [BW00, §VIII.5]. Then there exists a finite index torsion-free subgroup $\Gamma < \Gamma_0$ and a morphism $\phi\colon \Gamma \to \mathbb{Z}$ such that $N := \ker(\phi)$ is finitely generated but not hyperbolic, see [LP24], and also [IMM23] for related results. As a subgroup of Γ, the group N cannot contain a copy of $\mathbb{Z}^2$. Thus, there exists by construction a discrete and faithful representation of N in $\mathrm{PU}(n,1)$, but there is no discrete, faithful, and geometrically finite representation of N in $\mathrm{PU}(n,1)$.

2.1.4 Discrete subgroups with critical exponent equal to 2

In this section we give a proof of Theorem C, using the techniques developped in [CMW23]. It uses the function f defined in Lemma 2.5. We also outline a second proof, which involves the complete vector field $\mathfrak{X} = \nabla f$, called the *natural flow* in [CMW23].

Proof of Theorem C. Let Γ be a discrete and torsion-free subgroup of $\mathrm{PU}(n,1)$ with critical exponent $\delta = 2$ and assume that $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ contains a compact subvariety A of positive dimension. First, we remark that Γ is nonelementary, as a consequence of Proposition 2.7-(a) and (c). Thus Γ admits a Patterson–Sullivan measure $(\mu_x)_{x\in\mathbb{H}^n_{\mathbb{C}}}$. Fix a point $p \in \mathbb{H}^n_{\mathbb{C}}$ and, for all $\theta \in \partial\mathbb{H}^n_{\mathbb{C}}$, denote by $B_\theta := B_\theta(\cdot, p)$ the Busemann function at θ which vanishes at p. Let f be the Γ-invariant function defined on $\mathbb{H}^n_{\mathbb{C}}$ by $f(x) := -\ln\|\mu_x\|$. By Lemma 2.5, for all tangent vector v at a point $x \in \mathbb{H}^n_{\mathbb{C}}$, we have

$$i\partial\overline{\partial} f(v, Jv) \geq 0.$$

Let $\widetilde{A} \subset \mathbb{H}^n_{\mathbb{C}}$ be the lift of A, x be a regular point of $\widetilde{A}$ and v be a nonzero vector in $T_x\widetilde{A}$. Then the plurisubharmonic function $f|_{\widetilde{A}}$ is constant and consequently $i\partial\overline{\partial} f(v, Jv) = 0$. The inequality given by Lemma 2.5 is thus an equality for this vector v. Using that Patterson–Sullivan measures are supported on $\Lambda(\Gamma)$, one sees that this equality can only happen when

$$\forall\theta \in \Lambda(\Gamma), dB_\theta(v)^2 + dB_\theta(Jv)^2 = \|v\|^2.$$

Formula (1.16) together with the equality case in the Cauchy–Schwarz inequality imply that the above equality happens if and only if $v \in \mathbb{C}v_{x\theta}$ for all $\theta \in \Lambda(\Gamma)$ and all $v \in T_x\widetilde{A}$, where $v_{x\theta} \in T_x\mathbb{H}^n_{\mathbb{C}}$ is the unit vector at x pointing in the direction of θ. We deduce that A has dimension 1 and that

$$\forall\theta \in \Lambda(\Gamma),\ v_{x\theta} \in T_x\widetilde{A}.$$

Let D be the unique complex geodesic containing x for which $T_xD = T_x\widetilde{A}$. Then $\Lambda(\Gamma) \subset \partial D$, and hence the convex hull of $\Lambda(\Gamma)$ is contained in D. We deduce that Γ preserves D. Moreover $A \subset D/\Gamma$ by Proposition 1.12. To conclude, notice that D/Γ is a Riemann surface containing a compact subvariety of positive dimension, so D/Γ is compact and Γ is a complex Fuchsian group. □

Remark. We now outline a second proof of Theorem C, inspired by [CMW23, Theorem 1.5]. According to [CMW23, Lemma 2.2], the vector field $\mathfrak{X}$ defined by ∇f on $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is complete, and defines a smooth flow $(\phi_t)_{t\in\mathbb{R}}$. If $x \in \mathbb{H}^n_{\mathbb{C}}$ and $(Y_1, \dots, Y_k)$ is a k-frame of $T_x\mathbb{H}^n_{\mathbb{C}}$ which spans a subspace V, then the infinitesimal contraction rate of this frame by $(\phi_t)_{t\in\mathbb{R}}$ is given by the real number $\mathrm{tr}(\nabla df(x)|_V)$ (see [CMW23, Lemma 2.5] for a precise statement). For $x \in \mathbb{H}^n_{\mathbb{C}}$ and $\theta \in \partial\mathbb{H}^n_{\mathbb{C}}$, there is a real basis $(e_1, e_2, \dots, e_{2n})$ of $T_x\mathbb{H}^n_{\mathbb{C}}$ with $e_1 = v_{x\theta}$ the unit vector pointing in the direction of θ and $e_2 = Je_1$, in which the matrix of $L_\theta + J^*L_\theta$ is $\mathrm{Diag}(2-\delta, 2-\delta, 2, \dots, 2)$. In particular, if $\delta(\Gamma) = 2$ and $V \subset T_x\mathbb{H}^n_{\mathbb{C}}$ is a complex subspace, then

$$\mathrm{tr}(L_{\theta|V}) = \frac{1}{2}\mathrm{tr}\left((L_\theta + J^*L_\theta)|_V\right) \geq 0,$$

with equality if and only if V has complex dimension 1 and $V = \mathbb{C}v_{x\theta}$. Let $\widetilde{A}$ be the lift in $\mathbb{H}^n_{\mathbb{C}}$ of a compact subvariety of positive dimension $A \subset \mathbb{H}^n_{\mathbb{C}}/\Gamma$. Then we have for all regular point x of $\widetilde{A}$

$$\mathrm{tr}(\nabla df|_{T_x\widetilde{A}}) \geq \int_{\partial\mathbb{H}^n_{\mathbb{C}}} \mathrm{tr}(L_{\theta|T_x\widetilde{A}})d\mu_x(\theta) \geq 0.$$

If, for some regular point x of $\widetilde{A}$, the above inequality was strict, then ϕ_{-t} would contract A for sufficiently small $t > 0$, which would contradict the fact that A is a volume minimizer in its homology class. Fixing from now on a regular point x of $\widetilde{A}$, we deduce that A has dimension 1 and, since Patterson–Sullivan measures of Γ are supported on the limit set $\Lambda(\Gamma)$ of Γ, we get

$$\forall\theta \in \Lambda(\Gamma), v_{x\theta} \in T_x\widetilde{A}.$$

We conclude as in the first proof.

2.2 Quotients of higher rank Hermitian symmetric spaces (with C. Davalo)

Our main result in a higher rank setting is Theorem D, that we now restate. Recall that entropy was defined in Section 1.3.6 and the root associated with the Shilov boundary in Definition 1.44. This result generalizes in higher rank Lemma 2.5 for the first statement and Theorem C for the second one.

Theorem D. *Let $\rho\colon \Gamma \to G$ be a discrete and faithful representation of a torsion-free group Γ into a classical simple Hermitian Lie group G of noncompact type. Assume that the critical exponent $\delta(\rho(\Gamma))$ of $\rho(\Gamma)$ is nonzero. Fixing a maximal flat $\exp(\mathfrak{a})$ inside the symmetric space $\mathbb{X}$ associated with G, with $\mathfrak{a} \subset \mathfrak{p}$, let $\breve{a} \in \mathfrak{a}^*$ be the root associated with the Shilov boundary of $\mathbb{X}$ and let $h_\rho(\breve{a})$ denote the entropy of ρ associated with the root $\breve{a}$.*

1. *If $h_\rho(\breve{a}) < 1$, then $\rho(\Gamma)\backslash\mathbb{X}$ does not contain any compact analytic subvariety of positive dimension.*
2. *If $h_\rho(\breve{a}) = 1$ and $\rho(\Gamma)$ is nonelementary, then $\rho(\Gamma)\backslash\mathbb{X}$ contains a compact analytic subvariety of positive dimension if and only if Γ is the fundamental group of a surface and $\rho(\Gamma)$ preserves a totally geodesic complex disk in $\mathbb{X}$.*

The meaning of nonelementary in a higher rank setting has been given in Section 1.3.5. As already mentioned in the introduction, the conclusion in Theorem D-2 can be strengthened: we will prove that the totally geodesic complex disk preserved by $\rho(\Gamma)$ is a *diagonal disk*, that is the image of the diagonal by the totally geodesic embedding of a maximal polydisk (see Definition 2.13). We also recall from the introduction the following corollary, obtained by applying Theorem D to maximal representations ρ of surface groups, which were shown in [PSW23, Theorem 9.8] to have entropy $h_\rho(\breve{a})$ equal to 1.

Corollary E. *Let G be a simple Hermitian Lie group of noncompact type and $\mathbb{X}$ its associated symmetric space. Let $\rho\colon \pi_1(S_g) \to G$ be a maximal representation. Then $\rho(\Gamma)\backslash\mathbb{X}$ contains a compact analytic subvariety of positive dimension if and only if $\rho(\Gamma)$ preserves a diagonal disk of $\mathbb{X}$.*

The strategy used to prove Theorem D is an elaboration of the arguments used in the proof of Theorem C. We first present an independently interesting step in the proof, which consists in a lower bound on the Levi form of Busemann functions in higher rank and then we turn to the proof of Theorem D.

2.2.1 Busemann functions and plurisubharmonicity

The following result is based on the computation of the Hessian of Busemann functions for symmetric spaces by Davalo in [Dav23], see Proposition 1.47. Its proof also relies on Lemma 1.45, stated and proven in subsection 1.3.3.

Proposition F. *Let G be a simple Hermitian Lie group of noncompact type and $(\mathbb{X}, \omega)$ its associated symmetric space viewed as a Kähler manifold. Fixing a maximal flat $\exp(\mathfrak{a})$ of $\mathbb{X}$ with $\mathfrak{a} \subset \mathfrak{g}$ and a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, and let $\breve{a} \in \mathfrak{a}^*$ be the root associated with the Shilov boundary of $\mathbb{X}$.*

Let $w \in \mathfrak{a}^+$ be another unit vector, and let ξ_w be the point on the visual boundary of $\mathbb{X}$ associated with w. Choose a Busemann function $B_{\xi_w}\colon \mathbb{X} \to \mathbb{R}$ at the point ξ_w. Two such functions differ only by an additive constant, so their Levi form is independent of this choice. It satisfies the inequality:

$$i\partial\bar{\partial}B_{\xi_w} \geq \frac{\breve{a}(w)}{2}\omega.$$

Moreover, equality holds if and only if ξ_w belongs to the Shilov boundary of $\mathbb{X}$.

Proof of Proposition F. We begin by proving the equality when w is the unique unit vector $\breve{w} \in \mathfrak{a}^+$ pointing towards the Shilov boundary of $\mathbb{X}$. Recall that the vector $X \in \mathfrak{a}$ defined before

Lemma 1.45 also points to the Shilov boundary of $\mathbb{X}$ and that $\check{a}(X) = 2$. Consequently, $\check{w} = tX$ with $t = \frac{\check{a}(\check{w})}{2}$.

Let $z \in \mathfrak{p}$. Define $\mathfrak{p}^{(0)}, \mathfrak{p}^{(1)}$ and $\mathfrak{p}^{(2)}$ as in Lemma 1.45 and decompose accordingly z as $z = z^0 + z^1 + z^2$ with $z^k \in \mathfrak{p}^{(k)}$. Then, by Proposition 1.47, we have

$$\begin{aligned} D^2 B_{\xi_{\check{w}}}(z,z) &= \sum_{k=1}^{2} \sum_{\substack{\lambda \in \Sigma \\ |\lambda(X)|=k}} |\lambda(\check{w})| \|z^k\|^2 \\ &= t \sum_{k=1}^{2} k \|z^k\|^2 \\ &= \frac{\check{a}(\check{w})}{2} \left(\|z^1\|^2 + 2\|z^2\|^2 \right). \end{aligned}$$

Combining Proposition 1.47 with Lemma 1.45, we get in a similar fashion

$$D^2 B_{\xi_{\check{w}}}(Jz, Jz) = \frac{\check{a}(\check{w})}{2} \left(\|z^1\|^2 + 2\|z^0\|^2 \right).$$

Using formula (1.1), we obtain:

$$i\partial\overline{\partial} B_{\xi_{\check{w}}}(z, Jz) = \frac{\check{a}(\check{w})}{2} \left(2\|z^0\|^2 + 2\|z^1\|^2 + 2\|z^2\|^2 \right) = \frac{\check{a}(\check{w})}{2} \|z\|^2.$$

Now let $w \in \mathfrak{a}^+$ be an arbitrary unit vector. Recall that $\check{w} := tX$ with $t = \frac{\check{a}(\check{w})}{2}$. The element w decomposes as $w = sX + w^\perp$ with

$$s = \frac{\check{a}(w)}{\check{a}(\check{w})}$$

and $w^\perp \in \ker \check{a}$. Then, with the convention $B_0 = 0$, we have $B_{\xi_w} = sB_{\xi_{\check{w}}} + B_{\xi_{w^\perp}}$. Since Busemann functions are plurisubharmonic, we deduce that

$$i\partial\overline{\partial} B_{\xi_w} \geq s i\partial\overline{\partial} B_{\xi_{\check{w}}} = s\frac{\check{a}(\check{w})}{2}\omega = \frac{\check{a}(w)}{2}\omega.$$

Finally, suppose that $w^\perp$ is nonzero. Then, using the notation introduced before the proof of Lemma 1.45, it can be written as

$$w^\perp = \sum_{s=1}^{r-1} \alpha_s X_s,$$

with $\alpha_s \neq 0$ for some $s \in \{1, \dots, r-1\}$. Using Proposition 1.47, it is easy to see that

$$D^2 B_{w^\perp}(X_s, X_s) > 0,$$

and hence that

$$i\partial\overline{\partial} B_{\xi_w}(X_s, JX_s) > \frac{\check{a}(w)}{2} \|X_s\|^2.$$

□

2.2.2 Patterson–Sullivan measure in the direction of maximal growth and Levi form

Proof of Theorem D-1. Let $(\mu_x)_{x \in \mathbb{X}}$ be a Patterson–Sullivan family associated to Γ. Also let

$$\overline{\mu_x} := \frac{\mu_x}{\mu_x(\partial_{\text{vis}} \mathbb{X})}$$

be the associated probability measure, and let $S \subset \partial_{\mathrm{vis}}\mathbb{X}$ be the support of these measures. Recall from Section 1.3.6 that S is included in the G-orbit of some unit vector $w_{\max} \in \mathfrak{a}^+$. As in [CMW23], define a Γ-invariant function $f\colon \mathbb{X} \to \mathbb{R}$ by

$$f(x) = -\ln \mu_x(\partial_{\mathrm{vis}}\mathbb{X}) = -\ln \int_S e^{-\delta(\Gamma)B_\xi(x,o)}\, d\mu_o(\xi).$$

For $x \in \mathbb{X}$ and $v \in T_x\mathbb{X}$, the same computation as in rank one yields

$$i\partial\overline{\partial} f_x(v, Jv) \geq \delta(\Gamma) \int_S \left(i\partial\overline{\partial} B_\xi(v, Jv) - \frac{\delta(\Gamma)}{2} \left(dB_\xi(v)^2 + dB_\xi(Jv)^2\right)\right) d\overline{\mu_x}(\xi), \tag{2.1}$$

see the proof of Lemma 2.5. Then using Proposition F, for all $\xi \in S$, we have

$$i\partial\overline{\partial} B_\xi(v, Jv) \geq \frac{\breve{a}(w_{\max})}{2}\|v\|^2.$$

Moreover, Corollary 1.51 gives

$$\delta(\Gamma) \leq h_\Gamma(\breve{a})\breve{a}(w_{\max}).$$

Using Formula (1.16), we finally obtain

$$\begin{aligned} i\partial\overline{\partial} f_x(v, Jv) &\geq \delta(\Gamma) \left(\frac{\breve{a}(w_{\max})}{2} - \frac{\delta(\Gamma)}{2} \right) \|v\|^2 \\ &\geq \frac{1}{2}\delta(\Gamma)\breve{a}(w_{\max})\left(1 - h_\Gamma(\breve{a})\right)\|v\|^2. \end{aligned} \tag{2.2}$$

It follows from Proposition 1.49 that if $\breve{a}(w_{\max}) = 0$, then since $\psi_\Gamma(w_{\max}) = \delta(\Gamma) > 0$, we would have $h_\Gamma(\lambda) = +\infty$. Hence if $h_\Gamma(\breve{a}) < 1$ and $\delta(\Gamma) > 0$, then $\breve{a}(w_{\max}) > 0$ and we deduce that f is a strictly plurisubharmonic Γ-invariant function. Therefore $\Gamma\backslash\mathbb{X}$ does not contain any compact complex subvariety of positive dimension. □

2.2.3 Compact analytic subvarieties and complex disks

In this subsection, we prove Theorem D-(2). The proof is analogous, but more involved than its rank one equivalent, Theorem C. We first need the definition of a diagonal disk. Items 1,2 and 3 in Subsection 1.3.3 show that there exists an embedding of a maximal polydisk Δ^r inside $\mathbb{X}$, where r is the real rank of $\mathbb{X}$. All such polydisks are conjugate by an element of G.

Definition 2.13. [BIW09] A disk $\Delta \subset \mathbb{X}$ is called *diagonal* if it is the diagonal of a maximal complex polydisk embedded in $\mathbb{X}$.

Proof of Theorem D-(2). We assume that $h_\rho(\Gamma) = 1$ and that $\Gamma\backslash\mathbb{X}$ contains a compact analytic subvariety A of positive dimension. Let $\widetilde{A}$ be the lift of A inside $\mathbb{X}$, and choose a point x_1 in the regular part of $\widetilde{A}$, a unit vector $v_1 \in T_x\widetilde{A}$ and an element $\gamma \in \Gamma$. Call x_2 the element $x_2 = \gamma x_1 \in \widetilde{A}$ and $v_2 = \gamma_* v_1$. For $j \in \{1, 2\}$ and $w \in T_{x_j}\mathbb{X}$, we denote by $\xi(x_j, w) \in \partial_{\mathrm{vis}}\mathbb{X}$ the class in $\partial_{\mathrm{vis}}\mathbb{X}$ of the geodesic ray $t \in \mathbb{R}_+ \mapsto \exp_{x_j}(tw)$, and we define the circles $\mathcal{C}_j$ in $\partial_{\mathrm{vis}}\mathbb{X}$ by

$$\mathcal{C}_j := \{\xi(x_j, e^{i\theta}v_j) \mid \theta \in [0, 2\pi]\}.$$

Step 1. For $j \in \{1, 2\}$ and for μ_{x_j} almost every element $\xi \in S$, we have $\xi \in \mathcal{C}_j$.

Let f be the function constructed during the Proof of Theorem D-(1). Since this function is Γ-invariant and plurisubharmonic, we deduce that $f|_{\widetilde{A}}$ is constant. As a consequence

$$i\partial\overline{\partial} f_x(v, Jv) = 0,$$

and Inequality (2.2) is an equality. We deduce that for μ_x almost every element $\xi \in S$, there is equality in Inequality (1.16), which we recall is the following :

$$dB_\xi(v)^2 + dB_\xi(Jv)^2 = |\langle v, v_{x,\xi}\rangle|^2 \leq \|v\|^2.$$

This is possible if and only if $v_j \in \mathbb{C}v_{x_j\xi}$, where $v_{x_j\xi}$ is the unique unit vector at x_j pointing to ξ. This is equivalent to saying that $\xi \in \mathcal{C}_j$.

Step 2. For $j \in \{1,2\}$, each circle $\mathcal{C}_j$ is contained in a G-orbit Ω_j whose projectivization is not included in the face $\ker \breve{\alpha}$ opposite to the Shilov boundary. This is because $\breve{\alpha}(w_{\max}) > 0$, as noticed in the proof of Theorem D-1. As a consequence, there is a well-defined projection $\pi_j \colon \Omega_j \to \partial_{\mathrm{Shi}}\mathbb{X}$, see Equation (1.15). Then, we have

$$\pi_j(\mathcal{C}_j) = \{e^{i\theta}\pi_j(\xi(x_j, v_j)) \mid \theta \in [0, 2\pi]\}.$$

This a consequence of the fact that for every $\theta \in [0, 2\pi]$, there is an element $g_\theta \in G$ that fixes x and realises the multiplication by $e^{i\theta}$ in the visual boundary of $\mathbb{X}$. Thus

$$\pi_j(\xi(x_j, e^{i\theta}v)) = \pi_j(\xi(x_j, g_\theta v)) = g_\theta\pi_j(\xi(x_j, v)) = e^{i\theta}\pi_j(\xi(x_j, v)).$$

Step 3. Conclusion.
For $j \in \{1,2\}$, there exists by [BIW09, Proposition 3.4], a unique diagonal disk $D_j \subset \mathbb{X}$ whose boundary equals $\pi(\mathcal{C}_j)$ and thus contains μ_{x_j} almost all points of $\pi(S)$. By unicity, $\Delta_2 = \gamma\Delta_1$. Since μ_{x_1} and μ_{x_2} are absolutely continuous with respect to the other and S is not elementary, we deduce that the intersection of $\partial\Delta_1$ and $\partial\Delta_2$ contains at least three points. We deduce from [CØ03, Theorem 4.7][1] that $\Delta_1 = \Delta_2$. In other words, the disk Δ_1 is Γ-invariant. Since Γ also acts cocompactly on a complex subvariety of positive dimension, it has cohomology dimension at least 2 and hence its action on Δ_1 is cocompact. ☐

To conclude this section, we mention that together with Colin Davalo we plan to study in more detail the properties of the function f constructed in the proof of Theorem D, with the aim of showing that some higher rank quotients are Stein.

[1] Notice that in that article, which precedes the general introduction of the notion of tightness in [BIW09], tight holomorphic is used as a synonym for diagonally embedded.

Chapter 3

Toroidal compactifications of ball quotients

In this chapter, we provide the proof of Theorems G, H, I, J and K. We first recall the construction of toroidal compactifications of ball quotients, using the description of the ball given in Section 1.2. Then, Sections 3.2, 3.3, 3.4, 3.5 contain respectively the proofs of Theorems G, H, J, K. They use Section 3.1 and are independent of each other, except for Section 3.5 which relies on Section 3.4.

3.1 Preliminaries

3.1.1 Quotient of the ball by an element in the center of the Heisenberg group

In this section, we will use the notation introduced in Section 1.2.1.3. We first describe the quotient of $\mathbb{H}^n_{\mathbb{C}}$ by the unipotent parabolic group generated by a non-trivial element Z in the center of the Heisenberg group. Denote by

$$\widetilde{\mathcal{L}}\colon \mathbb{H}^n_{\mathbb{C}} \longrightarrow \widetilde{\Omega} := \{(a,u) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 2\Re(a) + \|u\|^2 < 0\}$$

the Siegel coordinates introduced in Section 1.2.1.3. Fix a basepoint $o = [f_2 - f_1] \in \mathbb{H}^n_{\mathbb{C}}$ and $\xi = [f_1] \in \partial\mathbb{H}^n_{\mathbb{C}}$. A computation similar to the proof of Lemma 1.19 shows that the point $n_{(v,s)} \cdot a_t \cdot o$ has coordinates

$$\widetilde{\mathcal{L}}\left(n_{(v,s)} \cdot a_t \cdot o\right) = \left(-\frac{\|v\|^2}{2} - is - e^{-2t}, v\right). \tag{3.1}$$

This shows in particular that the map $N_\xi \times \mathbb{R} \to \mathbb{H}^n_{\mathbb{C}}$ which associates with each element $(n_{(v,s)}, t)$ the point $n_{(v,s)} \cdot a_t \cdot o$ of the complex hyperbolic space is a diffeomorphism.

If Z is an element of N_ξ of the form $Z := n_{(0,l)}$ with $l \neq 0$, the group $\langle Z \rangle$ which it generates acts freely and properly discontinuously on $\mathbb{H}^n_{\mathbb{C}}$. The action of Z is expressed in the coordinates

$\widetilde{\mathcal{L}}$ by the translation $(a,u) \mapsto (a - il, u)$, thus the diagram of holomorphic maps

$$\begin{array}{ccc} \mathbb{H}^n_{\mathbb{C}} & \xrightarrow{\widetilde{\mathcal{L}}} & \widetilde{\Omega} \\ \downarrow & & \downarrow{\scriptstyle (\exp(\frac{2\pi \bullet}{l}), \mathrm{Id})} \\ \mathbb{H}^n_{\mathbb{C}}/\langle Z \rangle & \xrightarrow{\mathcal{L}} & \Omega. \end{array} \tag{3.2}$$

commutes, where the map $\mathcal{L}$ defined by the diagram is a biholomorphism on its image

$$\Omega := \left\{ (b,v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |b| < \exp\left(\frac{-\pi \|v\|^2}{l} \right) \right\}. \tag{3.3}$$

The action of Z preserves the horoballs centered at ξ, which are subset of $\mathbb{H}^n_{\mathbb{C}} \simeq N_\xi \times \mathbb{R}$ of the form $N_\xi \times (-\infty, t_0)$ for some real number t_0. A horoball centered at ξ will be denoted by H_ξ or simply H in what follows. In coordinates $\widetilde{\mathcal{L}}$ and $\mathcal{L}$ respectively, a horoball $N_\xi \times (-\infty, t_0)$ and its quotient by the action of Z identify with the open sets

$$\widetilde{\Omega}_{t_0} := \{(a,v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid \Re(a) < -\frac{\|v\|^2}{2} - \exp(-2t_0)\} \quad \text{and}$$
$$\Omega_{t_0} := \{(b,v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |b| < \lambda(t_0) \exp(\frac{-\pi \|v\|^2}{l})\},$$

with $\lambda(t_0) := \exp(\frac{-2\pi e^{-2t_0}}{l})$.

3.1.2 Structure of a noncompact finite-volume quotient of the ball

Let Γ_0 be a non-uniform torsion-free lattice in $\mathrm{PU}(n,1)$, and Γ be a finite index subgroup of Γ_0. Then by the Margulis lemma (see Theorem 1.37 or [BGS85]) the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a *thick-thin decomposition* with a finite number of cusps, which means that we can write

$$\mathbb{H}^n_{\mathbb{C}}/\Gamma = Q \cup \coprod_{i=1}^{k} C_i,$$

where Q is a compact subset of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, $k \in \mathbb{N}^*$, and for any $1 \leq i \leq k$, C_i is a finite-volume cusp, i.e., there exists a point ξ_i of the boundary of $\mathbb{H}^n_{\mathbb{C}}$ and a sufficiently deep horoball H_i centered at ξ_i, globally fixed by the stabiliser $\mathrm{Stab}_\Gamma(\xi_i)$ of ξ_i in Γ, such that C_i is biholomorphically isometric to $H_i/\mathrm{Stab}_\Gamma(\xi_i)$ (see Section 1.2.6.2 or [BGS85; SY82]). The group $\mathrm{Stab}_\Gamma(\xi_i)$ contains no hyperbolic elements, thus it is a subgroup of $N_{\xi_i} \rtimes K_i$, where K_i is the pointwise stabiliser in $\mathrm{PU}(n,1)$ of a geodesic ray ending at ξ_i. In particular, $\mathrm{Stab}_\Gamma(\xi_i)$ fixes each horosphere $N_{\xi_i} \times \{t\}$. Using the Auslander–Bieberbach theorem, one can show that there exists a finite subset $S \subset \Gamma_0$ such that for every finite index normal subgroup Γ of Γ_0 which does not intersect S, the parabolic elements of Γ are purely unipotent [Hum98]. In this case, $\mathrm{Stab}_\Gamma(\xi_i)$ is a subgroup of N_{ξ_i} for all i, and finally the cusp C_i is isomorphic to the product of the nilmanifold $N_{\xi_i}/(\Gamma \cap N_{\xi_i})$ with the interval $(-\infty, t_0)$ for some real number t_0 [Hum98]. Furthermore, since $H_i/(\Gamma \cap N_i)$ has finite volume, Fubini's theorem implies that $N_{\xi_i}/(\Gamma \cap N_{\xi_i})$ also has finite volume, and thus that $\Gamma \cap N_{\xi_i}$ is a lattice in N_{ξ_i}.

3.1.3 Construction of toroidal compactifications of ball quotients

We will say that Γ is a lattice for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification if Γ is non-uniform, torsion-free and all parabolic subgroups of Γ are purely unipotent, as explained in the previous section. This is justified by the following proposition, which, for such lattices Γ, allows one to compactify the manifold $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ by identifying each cusp with a holomorphic punctured disk bundle over a complex torus of dimension $n-1$. The compactification can then be done by adding the zero section of this bundle. This identification is classical (see for instance [Mok12, Section 2.1]), and we include it for the reader's convenience.

Proposition 3.1. *Let N be the $(2n-1)$-dimensional Heisenberg group, identified with N_ξ for some point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$, Λ a lattice in N, which acts on $\mathbb{H}^n_{\mathbb{C}} \simeq N \times \mathbb{R}$ by left multiplication on the N-factor, and let $H := N \times (-\infty, t_0)$ be a horoball of the complex hyperbolic space. Let $\rho\colon N \to \mathbb{C}^{n-1}$ be the surjective group morphism $n_{(v,s)} \mapsto v$. Then the map $\pi\colon H/\Lambda \to \mathbb{C}^{n-1}/\rho(\Lambda)$ is a holomorphic punctured disk bundle over the torus $T := \mathbb{C}^{n-1}/\rho(\Lambda)$, which has negative curvature. More precisely, there exists a holomorphic line bundle $\widetilde{\pi}\colon L \to T$ on the torus T, endowed with a Hermitian metric h with negative curvature $\Theta := -i\partial\bar{\partial}\log h$, as well as a holomorphic embedding $i\colon H/\Lambda \to L$ whose image is the open set:*

$$\{v \in L \mid 0 < \|v\|_h < 1\},$$

and such that $\pi = \widetilde{\pi} \circ i$.

Proof. Since Λ is a lattice in N, $\rho(\Lambda)$ is a lattice in $\mathbb{C}^{n-1}$ and in particular T is a torus. The discrete subgroup $\Lambda \cap [N, N]$ is non-trivial, generated by an element $Z := n_{(0,l)}$ for some $l > 0$. The map $\mathcal{L}$ defined above identifies $H/\langle Z\rangle$ with an open subset Ω_{t_0} of $\mathbb{C} \times \mathbb{C}^{n-1}$. Through this identification, Λ acts on Ω_{t_0} by

$$n_{(v,s)} \cdot (b, w) = \left(\exp\left(\frac{2\pi}{l}\left(-\frac{\|v\|^2}{2} - is - \langle w, v\rangle\right)\right) b, v + w\right).$$

The quotient space H/Λ can thus be identified with the quotient of Ω_{t_0} by this action. Notice that this action naturally extends to an action of Λ on $\mathbb{C} \times \mathbb{C}^{n-1}$. Let us endow the trivial line bundle map $\mathrm{pr}_2\colon \mathbb{C} \times \mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ with the Hermitian metric

$$h((b, v), (b', v)) := \lambda(t_0)^{-2} \exp\left(\frac{2\pi\|v\|^2}{l}\right) b\bar{b}',$$

so that the inclusion map $j\colon \Omega_{t_0} \to \mathbb{C} \times \mathbb{C}^{n-1}$ defines a holomorphic embedding whose image is the open set

$$\{(b, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < \|(b, v)\|_h < 1\}.$$

Since Ω_{t_0} is Λ-invariant, both the inclusion j and the bundle map pr_2 pass to the quotient, i.e. we have the following commutative diagram:

$$\begin{array}{ccc} \Omega_{t_0} & \longrightarrow & \Omega_{t_0}/\Lambda \\ \big\downarrow j & & \big\downarrow i \\ \mathbb{C} \times \mathbb{C}^{n-1} & \longrightarrow & L \\ \big\downarrow \mathrm{pr}_2 & & \big\downarrow \widetilde{\pi} \\ \mathbb{C}^{n-1} & \longrightarrow & T. \end{array}$$

In this diagram, L is the quotient of $\mathbb{C} \times \mathbb{C}^{n-1}$ by Λ, horizontal arrows are natural quotient maps and $\widetilde{\pi}$ is the only map making the bottom square of the diagram commute. Using the above expression for the action of Λ on $\mathbb{C} \times \mathbb{C}^{n-1}$, we see that $\widetilde{\pi}\colon L \to T$ is a line bundle and L can be endowed with the quotient Hermitian metric of h, still denoted by h, so that the image of i is the open set

$$\{v \in L \mid 0 < \|v\|_h < 1\}.$$

The curvature of (L, h), locally given by the form $-\frac{2\pi i}{l}\partial\bar{\partial}\|\cdot\|^2$, is negative. □

Let us denote by X_Γ the toroidal compactification of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ obtained by the previous construction.

Recall that a line bundle $L \to X$ on a complex manifold X is negative if it admits a Hermitian metric h whose curvature form is negative, i.e. if the Chern class $c_1(L)$ of the bundle L can be represented by a negative $(1,1)$-form. We define a *boundary torus of* X_Γ to be one of the tori added during the compactification. For each boundary torus T, the normal bundle of T in X_Γ is identified with the normal bundle $(TL/TO)|_O$ of the zero section O in the total space of the bundle L, which is isomorphic to L. We deduce the following corollary, which implies that boundary tori are in fact projective manifolds.

Corollary 3.2. *For each boundary torus $T \subset X_\Gamma$ added during the compactification, the normal bundle of T in X_Γ is negative.*

Remark. Let $n \geq 2$ and Γ be a non-uniform lattice in $\mathrm{PU}(n,1)$. Then it is known at least since [Mum75] that every holomorphic function $f\colon \mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{C}$ is constant. In particular, as mentioned in the first chapter, $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is not a Stein manifold. Indeed, the construction of toroidal compactifications together with the negativity of the normal bundle of the boundary tori show that there exists a finite index subgroup Γ_0 of Γ, a projective variety X realised inside some projective space $\mathbb{CP}^N$ with $N \geq 2$, containing a finite set S of singular points and a biholomorphism between $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ and $X \setminus S$. Let $f\colon \mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{C}$ be a holomorphic function, and lift it to a function $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0 \to \mathbb{C}$, seen as a function $\widetilde{f}\colon X \setminus S \to \mathbb{C}$. For every hyperplane H that contains x but does not intersect S, the holomorphic function $\widetilde{f}|_{X\cap H}$ is constant by the maximum principle. We deduce that $d\widetilde{f}(x)$ vanishes on a dense subset of T_xX. Thus $\widetilde{f}$ and hence f is constant.

3.2 Inexistence of a Kähler metric on X_Γ with nonpositive sectional curvature

In this section, we prove the following result, already stated in the introduction.

Theorem G. *Let Γ be a non-uniform and torsion-free lattice in $\mathrm{PU}(n,1)$ for which the quotient space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification X_Γ. Then there is no Kähler metric with nonpositive sectional curvature on X_Γ.*

We will use that X_Γ contains a complex torus whose normal bundle is negative (Corollary 3.2). We first state our curvature convention. On a Riemannian manifold (M, g), we define

$$\begin{aligned} R(X,Y) &:= \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] \qquad \text{and} \\ R(X,Y,Z,T) &:= g(R(X,Y)Z,T). \end{aligned} \tag{3.4}$$

Thus (M, g) has nonpositive sectional curvature if and only if $R(X,Y,X,Y) \leq 0$ for all vectors fields X, Y of M. Accordingly, if $(f_1, \dots, f_n)$ is an orthonormal basis of T_xM, the Ricci curvature of M at x is defined by

$$\mathrm{Ricci}_M(X,Y) := \sum_{j=1}^{n} R(X, f_j, Y, f_j).$$

Theorem G will be seen as a direct consequence of the more general Proposition 3.4. We will first need a useful lemma that is known since at least 1980 but has never been explicitely written as such.

3.2.1 An inequality for Kähler manifolds whose curvature have constant sign

The following Lemma can be traced back to Mostow and Siu [MS80] in the case of complex surfaces. It was used by Di Cerbo in [Di 12] to show that for every non-uniform lattice in $\mathrm{PU}(2,1)$ whose parabolic elements are unipotents, the toroidal compactification of $\mathbb{H}^2_{\mathbb{C}}/\Gamma$ has no Kähler metric of nonpositive sectional curvature.

Lemma 3.3 ([MS80, Lemma 1],[Di 12, Inequality (1)]). *Let (M,ω) be a Kähler manifold with either nonpositive or nonnegative Riemannian curvature. Then for all vectors v,w tangent to M at the same point, we have*

$$R(v,Jv,w,Jw)^2 \leq R(v,Jv,v,Jv)R(w,Jw,w,Jw). \tag{3.5}$$

Proof. The proof is parallel for both signs of the curvature, so we can assume that (M,ω) has nonpositive curvature. The proof consists in considering a one-parameter family of tangent planes at the considered point, and in expressing the curvature of these planes as a nonpositive function of the parameter. We use the fact that the metric is Kählerian to simplify the expression of the curvature. More precisely, for any real number a, we set

$$\begin{cases} p_a := av + w \\ q_a := J(av - w). \end{cases}$$

Since $p_{-a} = Jq_a$ and $q_{-a} = -Jp_a$, the map $a \mapsto R(p_a,q_a,p_a,q_a)$ is even. It can thus be written as a polynomial $P(a^2)$ of degree at most 2 in a^2:

$$P(a^2) = R(p_a,q_a,p_a,q_a) = R(v,Jv,v,Jv)a^4 + Aa^2 + R(w,Jw,w,Jw),$$

with

$$\begin{aligned} A :=& R(v,Jv,w,-Jw) + R(v,-Jw,v,-Jw) + R(v,-Jw,w,Jv)+ \\ & R(w,Jv,v,-Jw) + R(w,Jv,w,Jv) + R(w,-Jw,v,Jv) \\ =& -R(v,Jv,w,Jw) + R(v,-Jw,v,-Jw) - R(v,-Jw,v,-Jw)+ \\ & R(w,Jv,v,-Jw) - R(w,Jv,v,-Jw) - R(v,Jv,w,Jw) \\ =& -2R(v,Jv,w,Jw). \end{aligned}$$

Thus

$$P(a^2) = R(v,Jv,v,Jv)a^4 - 2R(v,Jv,w,Jw)a^2 + R(w,Jw,w,Jw).$$

Since the sectional curvature is nonpositive, the polynomial P is nonpositive on $\mathbb{R}_+$. Recall also that $R(X,JX,Y,JY) \leq 0$ for all vectors $X,Y \in T_xM$. We distinguish two cases:

- If the coefficient $R(v,Jv,v,Jv)$ vanishes, then when a becomes increasingly large, the inequality $R(p_a,q_a,p_a,q_a) \leq 0$ forces $R(v,Jv,w,Jw)$ to vanish as well.
- Otherwise, set

  $$a = \left(\frac{R(v,Jv,w,Jw)}{R(v,Jv,v,Jv)}\right)^{1/2}.$$

  Writing the nonpositivity of $P(a)$ gives the desired inequality.

In all cases, we get Inequality (3.5). □

3.2.2 Nonpositively curved Kähler manifolds containing submanifolds with $c_1 = 0$

The following proposition implies Theorem G. It uses the fact that the holomorphic sectional curvature decreases when passing to submanifolds, as well as relations between different notions of curvature, some of which are valid only for Kähler manifolds, and which are explained in [Div22].

Proposition 3.4. *Let M be a complex manifold with a Kähler metric of nonpositive sectional curvature. Suppose that M contains a compact submanifold V whose first Chern class $c_1(TV)$ is zero. Then V is totally geodesic and the Ricci curvature of M is zero in restriction to V. Moreover the first Chern class $c_1(N_V)$ of the normal bundle of V vanishes.*

Proof of Proposition 3.4. We are going to use the following results about curvature in Kähler geometry:

1. The holomorphic sectional curvature is non-increasing on submanifolds, and if it does not decrease, then the submanifold is totally geodesic, see [KN96, chapter IX, proposition 9.2].
2. In a compact Kähler manifold M, the total scalar curvature (i.e. the integral of its scalar curvature) is zero if the Chern class $c_1(TM)$ vanishes, see [Div22, remark 2.8].
3. In a compact Kähler manifold with nonpositive holomorphic sectional curvature, the vanishing of the total scalar curvature implies the vanishing of the holomorphic sectional curvature, see [Div22, Proposition 2.9].
4. A Kähler manifold with vanishing holomorphic sectional curvature has vanishing sectional curvature, see [KN96, chapter IX, proposition 7.1].

The holomorphic sectional curvature of V is less than or equal to that of M restricted to the tangent bundle of V by property 1. above, therefore, it is nonpositive. Using properties 2. and 3. above, we obtain that the holomorphic sectional curvature of V is identically zero. Hence the metric g restricted to V is flat by property 4. Finally the second part of property 1. implies that V is totally geodesic.

Let us now show that the Ricci curvature of M restricted to TV is identically zero. Let k be the dimension of V, and $(z^1, \dots, z^n)$ be complex coordinates in the neighborhood of a point x of V, such that $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n})$ forms an orthonormal basis of $T_x^{(1,0)}M$ and $(\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^k})$ an orthonormal basis of $T_x^{(1,0)}V$. Then, writing $z^i =: x^i + \sqrt{-1}y^i$ and $e_i := \frac{\partial}{\partial x^i}$, we have that

- $(e_1, Je_1, \dots, e_n, Je_n)$ is an orthonormal basis of the tangent space T_xM ;
- $(e_1, Je_1, \dots, e_k, Je_k)$ is an orthonormal basis of the tangent space T_xV.

Since e_1 can be any unitary vector at x tangent to V, it is enough to show that $\mathrm{Ricci}_M(e_1, e_1)$ vanishes. Recall that the first Bianchi identity gives for all vectors $X, Y \in T_xM$:

$$R(X, JX, Y, JY) = R(X, Y, X, Y) + R(X, JY, X, JY).$$

In particular, if the sectional curvature is nonpositive, so is the holomorphic bisectional curvature.

This formula also allows us to express the Ricci curvature at x

$$\begin{aligned}\mathrm{Ricci}_M(e_1,e_1) &= \sum_{j=1}^{n} R(e_1,e_j,e_1,e_j) + R(e_1,Je_j,e_1,Je_j)\\ &= \sum_{j=1}^{n} R(e_1,Je_1,e_j,Je_j)\\ &= \mathrm{Ricci}_V(e_1,e_1) + \sum_{j=k+1}^{n} R(e_1,Je_1,e_j,Je_j)\\ &= \sum_{j=k+1}^{n} R(e_1,Je_1,e_j,Je_j).\end{aligned}$$

We now apply Inequality (3.5) to $v = e_1$ and $w = e_j$ for all $j > k$: this shows that $\mathrm{Ricci}_{M\,|TV} = 0$ since $R(e_1, Je_1, e_1, Je_1) = 0$, due to the fact that the holomorphic sectional curvature of M restricted to TV is zero.

Let us now show that the normal bundle N_V of V has vanishing first Chern class. Multiplicativity of the total Chern classes with respect to the exact sequence

$$0 \longrightarrow TV \longrightarrow TM|_V \longrightarrow N_V \longrightarrow 0$$

yields the formula $c_1(N_V) = c_1(TM)|_V - c_1(TV)$. Moreover, if $\rho := \mathrm{Ricci}_M(\cdot, J\cdot)$ denotes the Ricci form of M, the class $c_1(TM)$ is represented by $-\frac{\sqrt{-1}}{2\pi}\rho$ [Kob87, Chapter 2], which vanishes in restriction to V. Since $c_1(TV) = 0$ by hypothesis, we deduce that $c_1(N_V) = 0$. □

3.3 A metric on X_Γ with nonpositive holomorphic bisectional curvature

This section is dedicated to the proof of Theorem H, that we restate below.

Theorem H. *Let Γ_0 be a non-uniform and torsion-free lattice in $\mathrm{PU}(n, 1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a Kähler metric with nonpositive holomorphic bisectional curvature.*

We will need the following lemma, whose proof we omit.

Lemma 3.5. *There exists a constant $C_0 > 0$ such that for every $C \geq C_0$, there is a smooth function $f\colon [0, C] \to \mathbb{R}_+$ which coincides with* cosh *on some neighborhood of* 0*, coincides with* exp *on some neighborhood of C and such that the functions f, f', f'', f''' are positive on $(0, C]$.*

The proof of Theorem H takes up Hummel and Schroeder's construction of a Kähler metric on toroidal compactifications [HS96], passing to finite index subgroups in order to get space for the construction of a metric with nonpositive holomorphic bisectional curvature. In a nutshell, the idea is to work separately in each cusp, and, provided that the cusp is "big enough", to patch there some compact manifold, endowed with a Kähler metric which glues to the complex hyperbolic one, and whose bisectional curvature is nonpositive. For the reader's convenience, we first provide a detailed description of the construction of the metric, (steps 1 to 5) as carried out by Hummel and Schroeder, and then we come to the original part of the proof, which is the computation of the holomorphic bisectional curvature (steps 6 and 7).

Step 1. Notation and description of a fixed cusp.

Using the model $(v,s) \mapsto n_{(v,s)}$ of the Heisenberg group N described in Section 1.2.3, we endow N with the left-invariant metric μ expressed at the identity point by

$$\mu_{\mathrm{Id}} := ds^2 \oplus dv^2.$$

Let $\mathfrak{n}$ be the Lie algebra of N, Z the generator of $[\mathfrak{n},\mathfrak{n}] \simeq \mathbb{R}$ such that $\exp(Z) = n_{(1,0)}$ and $\mathfrak{r}$ the μ-orthogonal of $[\mathfrak{n},\mathfrak{n}]$. Assuming that Γ_0 is a torsion-free non-uniform lattice in $\mathrm{PU}(n,1)$ with purely unipotent parabolic elements as explained in Section 3.1, any cusp in $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$ is biholomorphically isometric to

$$((-\infty, A_0] \times N/\Lambda_0, \mu_0, J_0),$$

where A_0 is some real number and Λ_0 a lattice in N which depend on the cusp; μ_0 is the N-invariant metric expressed at a point $(t, \mathrm{Id} \mod \Lambda_0)$ by

$$\mu_{0(t,\mathrm{Id} \mod \Lambda_0)} := dt^2 \oplus \exp(2t)\mu|_{\mathfrak{r}} \oplus \exp(4t)\mu|_{\mathbb{R}Z};$$

and J_0 is the N-invariant complex structure defined at a point $(t, \mathrm{Id} \mod \Lambda_0)$ by

$$\begin{cases} J_0 Z = \exp(2t)\frac{\partial}{\partial t}, \\ \forall v \in \mathbb{C}^{n-1}, \exp \circ J_0 \circ \exp^{-1}(n_{(0,v)}) = n_{(0,iv)}. \end{cases}$$

Step 2. Passing to a finite index subgroup of the lattice.

This is achieved in the following lemma, which is a direct consequence of Malcev's theorem.

Lemma 3.6. *Let Γ_0 be a non-uniform torsion-free lattice in $\mathrm{PU}(n,1)$, and C_0 be the constant given by Lemma 3.5. Then there exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, any cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, which lies above some cusp $((-\infty, A_0] \times N/\Lambda_0, \mu_0, J_0)$ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma_0$, is biholomorphically isometric to $((-\infty, A_0] \times N/\Lambda, \mu_0, J_0)$ for some finite index subgroup $\Lambda < \Lambda_0$ such that the group $\Lambda \cap [N,N]$ is generated by $n_{(l,0)}$ with $l > 2\pi \exp(2(C_0 - A_0))$.*

Step 3. Description of Hummel and Schroeder's model.

We now fix Γ_0 a non-uniform torsion-free lattice in $\mathrm{PU}(n,1)$ and Γ a finite index subgroup of Γ_0 as in Lemma 3.6. Let $((-\infty, A_0] \times N/\Lambda, \mu_0, J_0)$ be a cusp of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$, and $l > 2\pi \exp(2(C_0 - A_0))$ be the positive number such that $\Lambda \cap [N,N]$ is generated by $n_{(l,0)}$. Let φ be the automorphism of N given by

$$\varphi(n_{(v,s)}) := n_{\left(\sqrt{\frac{2\pi}{l}}v, \frac{2\pi}{l}s\right)}.$$

The next lemma describes the model used by Hummel and Schroeder to show that X_Γ is a Kähler manifold, and in which we will define a Kähler metric of nonpositive holomorphic bisectional curvature.

Lemma 3.7. *Set $C := \ln\sqrt{\frac{l}{2\pi}}$, $A := A_0 + C$, and let $f\colon [0,A] \to \mathbb{R}_+$ be the function given by Lemma 3.5 and $g := ff'$. Then $((-\infty, A_0] \times N/\Lambda, J_0)$ is biholomorphic to $((0,A] \times N/\varphi(\Lambda), J)$, where J is the N-invariant complex structure defined at a point $(t, Id \mod \varphi(\Lambda))$ by:*

$$\begin{cases} JZ = g(t)\frac{\partial}{\partial t}, \\ \forall v \in \mathbb{C}^{n-1}, \exp \circ J \circ \exp^{-1}(n_{(0,v)}) = n_{(0,iv)}. \end{cases}$$

Proof. First define the diffeomorphism

$$\Phi : \begin{cases} (-\infty, A_0] \times N/\Lambda & \longrightarrow & (-\infty, A] \times N/\varphi(\Lambda) \\ (t, n \mod \Lambda) & \longmapsto & (t + C, \varphi(n) \mod \varphi(\Lambda)). \end{cases}$$

Easy computations show that $\Phi_* \mu_0 = \mu_0$ and $\Phi_* J_0 = J_0$. In this new model, we have by construction $\varphi(n_{(l,0)}) = n_{(2\pi,0)}$ and $A > C_0$.

Then let $\psi \colon (0, A] \to (-\infty, A]$ be the solution of the following differential equation

$$\begin{cases} \psi'(t) = & \frac{\exp(2\psi(t))}{g(t)} \\ \psi(A) = & A. \end{cases}$$

It is easily seen that ψ is a well defined diffeomorphism, and that ψ is the identity in a neighborhood of A. Define

$$\Psi := \psi \times \mathrm{Id} \colon (0, A] \times N/\varphi(\Lambda) \longrightarrow (-\infty, A] \times N/\varphi(\Lambda).$$

A computation shows that $J := \Psi^* J_0$ is the N-invariant complex structure described in the Lemma. Thus Ψ provides the required biholomorphism. □

Step 4. A Kähler metric on the cusp.

We endow the complex manifold $((0, A] \times N/\varphi(\Lambda), J)$ with the N-invariant Hermitian metric $\mu_{f,g}$ expressed at a point $(t, \mathrm{Id} \mod \varphi(\Lambda))$ by

$$\mu_{f,g\ (t,\mathrm{Id} \mod \varphi(\Lambda))} := dt^2 \oplus f(t)^2 \mu|_{\mathfrak{r}} \oplus g(t)^2 \mu|_{\mathbb{R}Z}.$$

Notice that the metric $\mu_{f,g}$ coincides with $\Psi^* \mu_0$ on a neighborhood of the boundary of the manifold, hence it glues back to the hyperbolic metric on the thick part of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$. We are going to use the following facts from [HS96].

Lemma 3.8. *1. The toroidal compactification of the cusp can be realized in this model in the following way: using polar coordinates, we first identify $(0, A) \times N/\langle n_{(2\pi,0)} \rangle$ with $\mathbb{D}^*(0, A) \times \mathbb{C}^{n-1}$, where $\mathbb{D}^*(0, A)$ is the punctured disk in $\mathbb{C}$ with radius A; then we take the quotient by the action of $\varphi(\Lambda)$ [HS96, Proof of Theorem 7].*

2. The equality $g = ff'$ implies that the metric $\mu_{f,g}$ is Kählerian [HS96, Proof of Theorem 7].

3. The facts that $f = \cosh$ and that $g = ff'$ imply that $\mu_{f,g}$ extends to the compactification of the cusp [HS96, Lemma 3.8].

Step 5. Some preliminary curvature computations.

As in [HS96], we use the curvature convention given by Equation (3.4) and we see the Riemannian curvature R of $\mu_{f,g}$ as a symmetric form defined on bivectors. In the next lemma, which will be used to compute the holomorphic sectional curvature of $\mu_{f,g}$, we identify the tangent space of $(0, A) \times N/\varphi(\Lambda)$ at a point of the form $(t, \mathrm{Id} \mod \varphi(\Lambda))$ with $\mathbb{R} \oplus \mathfrak{n}$.

Lemma 3.9 ([HS96, Appendix])**.** *Let $X \in \mathfrak{r}$ be a tangent vector at $(t, Id \mod \varphi(\Lambda))$ which satisfies $\mu(X, X) = 1$. Define the bivectors*

$$(E_1, E_2, E_3, E_4, E_5, E_6) := \left(\frac{X \wedge JX}{\|X \wedge JX\|}, \frac{Z \wedge JZ}{\|Z \wedge JZ\|}, \frac{X \wedge Z}{\|X \wedge Z\|}, \frac{JX \wedge JZ}{\|JX \wedge JZ\|}, \frac{JX \wedge Z}{\|JX \wedge Z\|}, \frac{X \wedge JZ}{\|X \wedge JZ\|} \right).$$

Then

$$(R(E_i, E_j))_{1\leq i,j\leq 6} =: \begin{pmatrix} G & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix},$$

where

$$G = \begin{pmatrix} -4\left(\frac{f'}{f}\right)^2 & -2\frac{f''}{f} \\ -2\frac{f''}{f} & -3\frac{f''}{f} - \frac{f'''}{f'} \end{pmatrix} =: \begin{pmatrix} -h_1^2 & -2k \\ -2k & -h_2^2 \end{pmatrix},$$
$$F = \begin{pmatrix} -\frac{f''}{f} & -\frac{f''}{f} \\ -\frac{f''}{f} & -\frac{f''}{f} \end{pmatrix} =: \begin{pmatrix} -h_3^2 & -k \\ -k & -h_4^2 \end{pmatrix}.$$

The functions $h_i\colon (0, A) \to (0, +\infty)$ defined through G and F are well defined because of the positivity conditions for f and its derivatives. Here as in the sequel, the functions f, g, k, h_i shall implicitely be evaluated at t.

Step 6. Computation of the holomorphic bisectional curvature of $\mu_{f,g}$.

This is done in the following proposition.

Proposition 3.10. *Let Y, Ξ be two tangent vectors at $(t, Id \mod \varphi(\Lambda))$ for some $t \in (0, A)$. Write $Y =: aX + bZ + cJZ$ and $\Xi =: \alpha\widetilde{X} + \beta Z + \gamma JZ$ with $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$, and $X, \widetilde{X} \in \mathfrak{r}$ of norm $\mu(X, X) = \mu(\widetilde{X}, \widetilde{X}) = 1$. Also write $\widetilde{X} = dX + eJX + W$ with $d, e \in \mathbb{R}$ and W a vector μ-orthogonal to X and JX. Then the holomorphic bisectional curvature of Y and Ξ is given by*

$$-R(Y \wedge JY, \Xi \wedge J\Xi) = a^2\alpha^2 \left((d^2 + e^2)f^4h_1^2 + \frac{2g^2}{f^2}\|W\|^2\right) + h_2^2g^4(b^2 + c^2)(\beta^2 + \gamma^2) + \tag{3.6}$$
$$2f^2g^2h_3^2 \left(a^2(\beta^2 + \gamma^2) + \alpha^2(b^2 + c^2) + 2a\alpha(b\beta + c\gamma)\mu(X, \widetilde{X}) + 2a\alpha(c\beta - b\gamma)\mu(X, J\widetilde{X})\right).$$

Proof. We break down the proof of Proposition 3.10 into 5 steps.

Step 1. Decomposition of $Y \wedge JY$ and $\Xi \wedge J\Xi$.

We compute that

$$\begin{aligned} Y \wedge JY =& B_1 + B_2 + B_3 \qquad \text{with:} \\ & B_1 = a^2X \wedge JX + (b^2 + c^2)Z \wedge JZ, \\ & B_2 = -ac(X \wedge Z + JX \wedge JZ), \\ & B_3 = -ab(JX \wedge Z + (-X \wedge JZ)). \end{aligned}$$

In the same way, $\Xi \wedge J\Xi = \beta_1 + \beta_2 + \beta_3$ with β_i defined as B_i by replacing a, b, c by α, β, γ and X by $\widetilde{X}$.

Step 2. Computation of $R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X})$.

We compute that $\mathcal{R} := R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X})$ equals to

$$\begin{aligned}
\mathcal{R} &= R(X, JX, dX + eJX + W, -eX + dJX + JW) \\
&= (d^2+e^2)R(X, JX, X, JX) + d(R(X, JX, X, JW) + R(X, JX, W, JX)) + \\
&\quad e(R(X, JX, JX, JW) - R(X, JX, W, X)) + R(X, JX, W, JW) \\
&= (d^2+e^2)R(X, JX, X, JX) + 2R(X, JX, X, eW + dJW) + R(X, JX, W, JW) \\
&= -(d^2+e^2)f^4h_1^2 - 2(\frac{3g^2}{f^2} + f'^2)\langle JX, eW + dJW\rangle + R(X, JX, W, JW) \\
&= -(d^2+e^2)f^4h_1^2 + 0 + R(X, JX, W, JW).
\end{aligned}$$

To pass from the third to the fourth line, we used Lemma 3.9 and the computation of $R(X, JX)X$ done in [HS96, page 305]. Also

$$\begin{aligned}
R(X, JX, W, JW) &= \langle \nabla_{[X,JX]}W - \nabla_X\nabla_{JX}W + \nabla_{JX}\nabla_X W, JW\rangle \\
&= 2\langle \nabla_Z W, JW\rangle \qquad \text{because } \nabla_X W = \nabla_{JX}W = 0 \ \text{ and } [X, JX] = 2Z \\
&= \frac{-2g^2}{f^2}\|W\|^2 \qquad \text{from [HS96, (3.2)].}
\end{aligned}$$

In conclusion

$$-R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X}) = (d^2+e^2)f^4h_1^2 + \frac{2g^2}{f^2}\|W\|^2.$$

Step 3. Computation of $R(B_1, \beta_1)$.

We compute that

$$\begin{aligned}
R(B_1, \beta_1) &= a^2\alpha^2 R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X}) - 2ka^2(\beta^2+\gamma^2)\|X \wedge JX\|\|Z \wedge JZ\| - \\
&\quad 2k\alpha^2(b^2+c^2)\|\widetilde{X} \wedge J\widetilde{X}\|\|Z \wedge JZ\| - h_2^2(b^2+c^2)(\beta^2+\gamma^2)\|Z \wedge JZ\|^2 \\
&= a^2\alpha^2 R(X \wedge JX, \widetilde{X} \wedge J\widetilde{X}) - 2h_3^2f^2g^2\left(a^2(\beta^2+\gamma^2) + \alpha^2(b^2+c^2)\right) - \\
&\qquad h_2^2g^4(b^2+c^2)(\beta^2+\gamma^2) \\
&= -a^2\alpha^2\left((d^2+e^2)f^4h_1^2 + \frac{2g^2}{f^2}\|W\|^2\right) - 2h_3^2f^2g^2\left(a^2(\beta^2+\gamma^2) + \alpha^2(b^2+c^2)\right) - \\
&\qquad h_2^2g^4(b^2+c^2)(\beta^2+\gamma^2).
\end{aligned}$$

Step 4. Computation of $R(B_i, \beta_j)$ for $i, j \in \{2, 3\}$

First notice that the map $(U, \widetilde{U}) \mapsto R(U \wedge Z, \widetilde{U} \wedge Z)$ is bilinear symmetric, of quadratic form

$$R(U \wedge Z, U \wedge Z) = -f^2g^2h_3^2\mu(U, U).$$

By splitting, we deduce that for all $U, \widetilde{U}$, we have $R(U \wedge Z, \widetilde{U} \wedge Z) = -f^2g^2h_3^2\mu(U, \widetilde{U})$. Then, using that the metric is Kählerian, we find that

$$\begin{aligned}
R(B_2,\beta_2) &= ac\alpha\gamma R(X\wedge Z + JX\wedge JZ, \widetilde{X}\wedge Z + J\widetilde{X}\wedge JZ)\\
&= 4ac\alpha\gamma R(X\wedge Z, \widetilde{X}\wedge Z)\\
&= -4f^2g^2h_3^2\mu(X,\widetilde{X})ac\alpha\gamma.\\
R(B_3,\beta_3) &= 4ab\alpha\beta R(JX\wedge Z, J\widetilde{X}\wedge Z)\\
&= -4f^2g^2h_3^2\mu(JX,J\widetilde{X})ab\alpha\beta\\
&= -4f^2g^2h_3^2\mu(X,\widetilde{X})ab\alpha\beta\\
R(B_2,\beta_3) &= 4ac\alpha\beta R(X\wedge Z, J\widetilde{X}\wedge Z)\\
&= -4f^2g^2h_3^2\mu(X,J\widetilde{X})ac\alpha\beta.\\
R(B_3,\beta_2) &= 4ab\alpha\gamma R(JX\wedge Z, \widetilde{X}\wedge Z)\\
&= -4f^2g^2h_3^2\mu(JX,\widetilde{X})ab\alpha\gamma\\
&= -4f^2g^2h_3^2\mu(X,J\widetilde{X})ab\alpha\gamma.
\end{aligned}$$

Step 5. Vanishing of the remaining terms.

According to the computations of the appendix of [HS96], $R(X,JX)Z$ is a multiple of $JZ = g\frac{\partial}{\partial t}$, which is orthogonal to both $\widetilde{X}$ and $J\widetilde{X}$. Therefore the two terms $R(B_1,\beta_2)$ and $R(B_1,\beta_3)$ vanish. By symmetry, we also have $R(B_2,\beta_1) = R(B_3,\beta_1) = 0$.

Putting *Steps* 2 to 5 together, we obtain Formula (3.6). □

Step 7. Conclusion: nonpositivity of the holomorphic bisectional curvature.

To show that $R(Y\wedge JY, \Xi\wedge J\Xi)\le 0$, it is enough to show that the term

$$a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2)+2a\alpha(b\beta+c\gamma)\mu(X,\widetilde{X})+2a\alpha(c\beta-b\gamma)\mu(X,J\widetilde{X})$$

in Formula (3.6) is nonpositive. This is a consequence of the following lemma.

Lemma 3.11.

$$\left|2a\alpha\left((b\beta+c\gamma)\mu(X,\widetilde{X})+(c\beta-b\gamma)\mu(X,J\widetilde{X})\right)\right|\le a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2).$$

Proof. The two real numbers $x := \mu(X,\widetilde{X})$ and $y := \mu(X,J\widetilde{X})$ verify the inequality

$$x^2+y^2 = |x+iy|^2 = |\mu_{\mathbb{C}}(X,\widetilde{X})|^2 \le \mu(X,X)\mu(\widetilde{X},\widetilde{X})\le 1,$$

where $\mu_{\mathbb{C}}$ is the hermitian product associated with $\mu|_{\mathfrak{r}}$, defined by

$$\mu_{\mathbb{C}}(U,\widetilde{U}) := \mu(U,\widetilde{U}) + i\mu(U,J\widetilde{U}).$$

Using the triangle inequality and the inequality $2XY\le X^2+Y^2$, we get that

$$\begin{aligned}
|2a\alpha\left((b\beta+c\gamma)x+(c\beta-b\gamma)y\right)| &\le 2|a\alpha||b\beta x+c\beta y| + 2|a\alpha||c\gamma x - b\gamma y|\\
&\le a^2\beta^2+\alpha^2(bx+cy)^2+a^2\gamma^2+\alpha^2(cx-by)^2\\
&\le a^2(\beta^2+\gamma^2)+\alpha^2(b^2+c^2).
\end{aligned}$$

□

This allows us to conclude that $R(Y \wedge JY, \Xi \wedge J\Xi)$ is nonpositive. Easy case by case examinations show that it vanishes only if $Y = 0$ or $\Xi = 0$. In conclusion, $\mu_{f,g}$ has negative holomorphic bisectional curvature on the cusp, and thus nonpositive holomorphic bisectional curvature on the compactified cusp. Gluing these metrics to the complex hyperbolic one in the thick part of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ gives a Kähler metric on X_Γ of nonpositive holomorphic bisectional curvature. This concludes the proof of Theorem H.

Remark. In fact the metric constructed in the proof has negative Ricci curvature. The Ricci curvature, being a sum of holomorphic bisectional curvatures, is negative on the open subset $\mathbb{H}^n_{\mathbb{C}}/\Gamma \subset X_\Gamma$, and we will prove that it is negatively pinched near the boundary tori. More precisely, using the notation of the proof of Theorem H, we claim that Ricci $\leq -2\mu_{f,g}$ on $(0,\epsilon) \times N/\phi(\Lambda)$ for some $\epsilon > 0$. Indeed, let $(X_1, \dots, X_{n-1})$ be an orthogonal $\mathbb{C}$-basis of $(\mathfrak{r}, \mu|_{\mathfrak{r}})$, so that, implictely evaluating f and $g = ff'$ at some real number $t \in (0, A]$, the family

$$\left(\frac{1}{f}X_1, \frac{1}{f}JX_1, \dots, \frac{1}{f}X_{n-1}, \frac{1}{f}JX_{n-1}, \frac{1}{g}Z, \frac{1}{g}JZ\right)$$

forms an orthonormal basis of the tangent space of $(0, A] \times N/\phi(\Lambda)$ at $(t, \mathrm{Id} \mod \varphi(\Lambda))$. Let

$$\Xi = \sum_{i=1}^{n-1}(\alpha_i X_i + \alpha_{\bar{i}} JX_i) + \beta Z + \gamma JZ$$

be a tangent vector at this point, with $\alpha_i, \alpha_{\bar{i}}, \beta, \gamma \in \mathbb{R}$, and denote by α the real number

$$\alpha := \sqrt{\sum_{i=1}^{n-1}(\alpha_i^2 + \alpha_{\bar{i}}^2)}.$$

Using the above formula for the bisectional curvature, a computation shows that the Ricci curvature of the metric has the following expression:

$$\mathrm{Ricci}(\Xi, \Xi) = -\left(2ff'' + 4f'^2 + 2(n-2)f'^2\right)\alpha^2 - \left((2n+1)ff'^2f'' + f^2f'f'''\right)(\beta^2 + \gamma^2).$$

The function f coincides with cosh on $(0, \epsilon)$, for some $\epsilon > 0$ and when $t < \epsilon$, we get

$$\mathrm{Ricci}(\Xi, \Xi) \leq -2f^2\alpha^2 - 2g^2(\beta^2 + \gamma^2) = -2\|\Xi\|^2_{\mu_{f,g}}.$$

By continuity, this inequality remains valid on the boundary tori, hence the result. However, notice that the riemannian sectional curvature of this metric cannot be nonpositive because of Proposition 3.4.

3.4 Commensurators, parabolic subgroups and Albanese maps

From now on, we work with arithmetic lattices. In order to get some immersivity properties for the Albanese map of toroidal compactifications, we will use results on the density of the commensurator of an arithmetic lattice inside some stabilizers, that we first explain, before passing to the proof of Theorem J.

3.4.1 Density of the commensurator of an arithmetic lattice inside the fixator of a generic geodesic

We begin this section by giving a definition and stating a proposition which will be used during the proof of Theorem J.

Definition 3.12. For $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ and $o \in \mathbb{H}^n_{\mathbb{C}}$, we define $K(\xi, o)$ as the subgroup of all elements in $\mathrm{PU}(n,1)$ fixing both ξ and o.

We first recall some facts about the group $K(\xi, o)$ that will be needed later. First this group is isomorphic to $\mathrm{U}(n-1)$. Moreover, recall from Section 1.2.1.3 that N_ξ is the unipotent radical of the stabilizer of ξ in $\mathrm{PU}(n,1)$. As explained in Section 3.1, if Z_ξ is a non-trivial element in the centre of N_ξ, then $\mathbb{H}^n_{\mathbb{C}}/\langle Z_\xi\rangle$ is biholomorphic to the open subset Ω of $\mathbb{C}^* \times \mathbb{C}^{n-1}$ defined by Formula (3.3). Define also

$$\widehat{\Omega} := \Omega \cup (\{0\} \times \mathbb{C}^{n-1}).$$

It can be verified that the action of $K(\xi, o)$ on $\mathbb{H}^n_{\mathbb{C}}$ commutes with $\langle Z_\xi\rangle$ so $K(\xi, o)$ also acts on Ω. Identifying $K(\xi, o)$ with $\mathrm{U}(n-1)$, this action can simply be written $A \cdot (\lambda, v) = (\lambda, Av)$ for all $(\lambda, v) \in \Omega \subset \mathbb{C}^* \times \mathbb{C}^{n-1}$. This follows from Formula (3.1) and Diagram (3.2), and from the fact that $K(\xi, o)$ acts only on the "v-terms" of the coordinates $\widetilde{\mathcal{L}}$ and preserves its norm. In particular, the action of $K(\xi, o)$ extends by the same formula to $\widehat{\Omega}$.

We now turn to Proposition 3.13. This proposition is probably classical, and we include it for the sake of being self-contained. If Γ is a discrete subgroup of $\mathrm{PU}(n,1)$, recall that its commensurator $\mathrm{Comm}(\Gamma)$ is the set of elements $g \in \mathrm{PU}(n,1)$ such that $g\Gamma g^{-1} \cap \Gamma$ has finite index both in Γ and in $g\Gamma g^{-1}$; it is a subgroup of $\mathrm{PU}(n,1)$. Moreover a parabolic point of Γ is a point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$ whose stabiliser $\mathrm{Stab}_\Gamma(\xi)$ in Γ contains a parabolic element.

Proposition 3.13. *Let Γ_0 be a non-uniform arithmetic lattice in* $\mathrm{PU}(n,1)$. *Then there exists a dense subset $S \subset \mathbb{H}^n_{\mathbb{C}}$ such that for any parabolic point ξ of Γ_0 and for any point $o \in S$,* $\mathrm{Comm}(\Gamma_0) \cap K(\xi, o)$ *is dense in* $K(\xi, o)$.

Proof of Proposition 3.13. We use here the explicit description of non-uniform arithmetic lattices in $\mathrm{PU}(n,1)$, see for instance [ES14, Section 2] or [PY09]. We denote $\pi\colon \mathrm{U}(n,1) \to \mathrm{PU}(n,1)$ and $[\cdot]\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ the projection maps. Let Γ_0 be a non-uniform arithmetic lattice in $\mathrm{PU}(n,1)$. Then there exists:

- A purely imaginary quadratic extension $l := \mathbb{Q}(\sqrt{-d})$ of $\mathbb{Q}$ whose ring of integers is denoted $\mathcal{O}_l$,
- A Hermitian form $H\colon l^{n+1} \times l^{n+1} \to l$ on l^{n+1} whose extension $H_{\mathbb{C}}$ to $\mathbb{C}^{n+1}$ is of signature $(n,1)$,
- An isomorphism $\Phi\colon (\mathbb{C}^{n+1}, H_{\mathbb{C}}) \to (\mathbb{C}^{n+1}, \langle\cdot,\cdot\rangle)$ where $\langle\cdot,\cdot\rangle$ denotes the standard Hermitian form of signature $(n,1)$ used to define $\mathbb{H}^n_{\mathbb{C}}$; moreover Φ induces an isomorphism $\Theta\colon \mathrm{U}(H_C) \to \mathrm{U}(n,1)$ defined by $\Theta(g) := \Phi g \Phi^{-1}$, and allows to define an arithmetic lattice $\Gamma_l < \mathrm{PU}(n,1)$ by $\Gamma_l := \pi(\Theta(U(H_C, \mathcal{O}_l)))$; and
- A subgroup Γ of $\Gamma_0 \cap \Gamma_l$ which is of finite index both in Γ_0 and Γ_l.

If we replace Γ by a finite index subgroup, we can also suppose that all its parabolic subgroups are unipotent. We define:

$$\mathbb{H}^n_l := \{[\Phi(x)] \mid x \in l^{n+1} \text{ and } H(x,x) < 0\},$$
$$\partial\mathbb{H}^n_l := \{[\Phi(x)] \mid x \in l^{n+1} \text{ and } H(x,x) = 0\}.$$

Let $\xi := [\Phi(\widetilde{\xi})] \in \partial\mathbb{H}_l^n$ and $o := [\Phi(\widetilde{o})] \in \mathbb{H}_l^n$, with $(\widetilde{\xi}, \widetilde{o}) \in l^{n+1} \times l^{n+1}$. Let C be the intersection of the stabilisers of $\widetilde{\xi}$ and $\widetilde{o}$ in $U(H_{\mathbb{C}})$, and C_l the subgroup of C consisting of the matrices with coefficients in l. By construction, $K(\xi, o) = \pi(\Theta(C))$.

Two commensurable lattices have the same commensurator and the same parabolic points. Therefore, the following three statements lead to the proposition with $S := \mathbb{H}_l^n$.

Assertion 1: let ξ be a parabolic point of Γ. Then $\xi \in \partial\mathbb{H}_l^n$.

Assertion 2: let $\xi \in \partial\mathbb{H}_l^n$ and $o \in \mathbb{H}_l^n$. Then $\pi(\Theta(C_l)) \subset \mathrm{Comm}(\Gamma_l) \cap K(\xi, o)$, with C_l defined as above.

Assertion 3: let $\xi \in \partial\mathbb{H}_l^n$ and $o \in \mathbb{H}_l^n$. Then C_l is dense in C, so $\pi(\Theta(C_l))$ is dense in $K(\xi, o)$.

The proofs of these three statements are independent of each other. All these assertions are very classical. We give elementary proofs of Assertions 1 and 3, though shorter and more general proofs exist.

Proof of Assertion 1 (see also [Zin79]).

Let γ be a non-trivial parabolic element of Γ which fixes $\xi = [x]$. We must show that $\mathbb{C}x$ intersects non trivially l^{n+1}. Since $\Gamma \subset \Gamma_l$, we can write $\gamma = \pi(\Theta(M))$ with $M \in \mathrm{U}(H_{\mathbb{C}}, \mathcal{O}_l)$, and $\Theta(M)x = \alpha x$ for some $\alpha \in S^1$. Since γ is unipotent, it can also be represented by an upper triangular matrix T with diagonal entries 1, thus $\Theta(M)$ is a multiple $\alpha' T$ of T. Evaluating at x gives $\alpha = \alpha'$, and in particular $\det(M) = \det(\Theta(M)) = \alpha^{n+1}$. We deduce that $\alpha^{n+1} \in \mathcal{O}_l$. The only integers of modulus 1 of purely imaginary quadratic extensions are roots of units, so α is a root of unit. Up to replacing γ by an appropriate power γ^k, we can thus suppose that $\gamma = [\Theta(M)]$ with $\Theta(M)x = x$.

The map M can also be seen as a linear map $M_l \colon l^{n+1} \to l^{n+1}$. The kernel E_l^1 of $M_l - \mathrm{Id}$ and the kernel E^1 of $M - \mathrm{Id}$ are related by $E^1 = E_l^1 \otimes \mathbb{C}$. Let $e := (e_1, \ldots, e_k)$ be an orthogonal basis of E_l^1. It is not possible to have $H(e_i, e_i) > 0$ for all i because otherwise, $H_{\mathbb{C}}$ would be positive definite on the $\mathbb{C}$-span of e, which is E^1, and this space contains the isotropic vector x. Thus $H(e_i, e_i) \leq 0$ for some i. Now $\xi = [x]$ and $[\Phi(e_i)]$ are two fixed points in $\overline{\mathbb{H}_{\mathbb{C}}^n}$ of the parabolic element $\gamma = \pi(\Theta(M))$, hence $\xi = [\Phi(e_i)] \in \partial\mathbb{H}_l^n$.

Proof of Assertion 3.

Assertion 3 is a consequence of the fact that C_l, being an algebraic group defined over $\mathbb{Q}$, is unirational over $\mathbb{Q}$ [Bor91, Theorem 18.2]. We now give a self-contained proof of this assertion.

Let $\widetilde{\xi} \in l^{n+1}$ and $\widetilde{o} \in l^{n+1}$ be respectively isotropic and negative vectors of H, P the l-span of these two vectors and $P^\perp$ its orthogonal with respect to H. Then, since H has signature $(n, 1)$, $H(\widetilde{\xi}, \widetilde{o}) \neq 0$ and this easily implies that $P \cap P^\perp = \{0\}$. Hence $l^{n+1} = P \oplus P^\perp$, and H restricted to $P^\perp$ is a positive definite Hermitian form. In particular, there is a basis $(u_1, \ldots, u_{n-1})$ of $P^\perp$ orthogonal for H, and the matrix $B := (H(u_i, u_j))_{i,j}$ is diagonal, with positive rational diagonal entries. Now define

$$\mathrm{U} := \{M \in \mathrm{Mat}(n-1, \mathbb{C}) \mid^t MB\overline{M} = B\}.$$

Then C is isomorphic to U by a Lie group isomorphism sending C_l to:

$$\mathrm{U}_l := \{M \in \mathrm{Mat}(n-1, l) \mid^t MB\overline{M} = B\}.$$

We will show that U_l is dense in U, using a classical argument involving the Cayley transform

[Wey46, section II.10]. Here are the notation that we are going to use:

$$\begin{array}{rr} \mathrm{Mat}^* := \{M \in \mathrm{Mat}(n-1,\mathbb{C}) \mid \det(M+\mathrm{I}) \neq 0\} & \mathrm{Mat}^*_l := \mathrm{Mat}^* \cap \mathrm{Mat}(n-1,l) \\ \mathrm{U}^* := \mathrm{U} \cap \mathrm{Mat}^* & U^*_l := \mathrm{U}^* \cap \mathrm{Mat}(n-1,l) \\ \mathrm{A}^* := \{M \in \mathrm{Mat}(n-1,\mathbb{C}) \mid^t MB = -B\overline{M}\} \cap \mathrm{Mat}^* & \mathrm{A}^*_l := \mathrm{A}^* \cap \mathrm{Mat}(n-1,l) \end{array}$$

The Cayley transform is the involution

$$\begin{array}{rccc} S\colon & \mathrm{Mat}^* & \longrightarrow & \mathrm{Mat}^* \\ & N & \longmapsto & 2(\mathrm{I}+N)^{-1} - \mathrm{I}. \end{array}$$

It is easily verified that $S\colon \mathrm{U}^* \to \mathrm{A}^*$ is a homeomorphism sending U^*_l to A^*_l. Since U^* is dense in U, it is thus enough to show that A^*_l is dense in A^*. This follows from the fact that A^* is the intersection of a dense open set and a subspace defined by rational equations. □

3.4.2 Proof of Theorem J

We now come to the proof of Theorem J. For every closed Kähler manifold X, one can construct its Albanese torus $\mathrm{Alb}(X)$ and a holomorphic map from X to $\mathrm{Alb}(X)$, called the Albanese map of X [Voi02, chapter 12]. We denote by $\mathrm{Alb}(X_\Gamma)$ the Albanese torus of X_Γ and $A_\Gamma\colon X_\Gamma \to \mathrm{Alb}(X_\Gamma)$ its Albanese map. With this notation, let us restate Theorem J.

Theorem J. *Let Γ_0 be a non-uniform and torsion-free arithmetic lattice in $\mathrm{PU}(n,1)$. There exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the toroidal compactification X_Γ of $\mathbb{H}^n_\mathbb{C}/\Gamma$ satisfies the following properties:*

- *Its Albanese map A_Γ is an immersion on the open subset $\mathbb{H}^n_\mathbb{C}/\Gamma \subset X_\Gamma$.*
- *For any boundary torus T of X_Γ, the map $A_{\Gamma|T}$ is either an immersion or a constant.*

We recall that by "boundary torus of X_Γ", we mean one of the tori added during the compactification, i.e. a connected component of the complement of $\mathbb{H}^n_\mathbb{C}/\Gamma$ in X_Γ. We also mention that in dimension $n \geq 3$, the boundary tori of the toroidal compactification X_Γ are never simple when the lattice Γ is arithmetic, so that the second point of Theorem J is not automatic: see the remark at the end of this section.

Proof. For the first point, we refer to [Eys18, Theorem 1.3], which uses that there are non-trivial holomorphic one forms on X_Γ by results of Shimura [Shi79]. See also [LP24, Section 3.1] for another exposition of this proof in the cocompact case. For the proof of the second point, we can and do assume that $\mathbb{H}^n_\mathbb{C}/\Gamma_0$ admits a toroidal compactification, and we introduce the following terminology: a boundary torus τ of X_{Γ_0} will be said to be *virtually Albanese non-constant* if there exists a finite index subgroup $\Gamma < \Gamma_0$ and a boundary torus T of X_Γ above τ, in restriction to which A_Γ is not constant. Let τ be a virtually Albanese non-constant boundary torus of X_{Γ_0}, and Γ and T be as above. Holomorphic maps between tori have constant rank, thus for every point $x \in T$, there exists a holomorphic one form $\alpha \in \Omega^1(X_\Gamma)$ such that $i^*\alpha(x) \neq 0$, where $i\colon T \hookrightarrow X_\Gamma$ is the inclusion.

Let $\xi \in \partial\mathbb{H}^n_\mathbb{C}$ be a point corresponding to the cusp of $\mathbb{H}^n_\mathbb{C}/\Gamma$ compactified by the torus T. From here on, we use the notation of Section 3.1. According to the results recalled there, $\mathrm{Stab}_\Gamma(\xi)$ is a lattice in the nilpotent group N_ξ. Thus it intersects $[N_\xi, N_\xi] \simeq \mathbb{R}$ non-trivially. Let Z_ξ be a generator of $\Gamma \cap [N_\xi, N_\xi]$. We have the following diagram.

$$\begin{array}{ccccccc} \mathbb{H}^n_{\mathbb{C}}/\langle Z_\xi\rangle & \xrightarrow{\mathcal{L}} & \Omega & \hookrightarrow & \widehat{\Omega} & \xhookleftarrow{\widetilde{i}} & \mathbb{C}^{n-1} \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{H}^n_{\mathbb{C}}/\Gamma & & \hookrightarrow & & X_\Gamma & \xhookleftarrow{i} & T, \end{array}$$

where $\widehat{\Omega} := \Omega \cup (\{0\} \times \mathbb{C}^{n-1})$, hooked arrows are inclusions, $\widetilde{i}$ is defined by $\widetilde{i}(x) := (0, x)$, and the map $\widehat{\Omega} \to X_\Gamma$ is a covering above $T \cup \mathbb{H}^n_{\mathbb{C}}/\Gamma$. Recall that $\mathcal{L}$ depends on the choice of a geodesic γ such that $\gamma(t) \underset{t\to+\infty}{\longrightarrow} \xi$. Here we choose γ such that $o := \gamma(0)$ belongs to the set S of Proposition 3.13. As explained in the beginning of this section, the group $K(\xi, o)$ acts on $\widehat{\Omega}$.

Let $x \in X_\Gamma$ be the point below $(0, 0) \in \widehat{\Omega}$, and $\alpha \in \Omega^1(X_\Gamma)$ a holomorphic one form such that $i^*\alpha(x) \neq 0$. Denoting by β the pullback of α by the map $\widehat{\Omega} \to X_\Gamma$, we have $\widetilde{i}^*\beta(0) \neq 0$. We deduce that there exist $g_1, \dots, g_{n-1} \in K(\xi, o)$ such that

$$e := (\widetilde{i}^*(g_1^{-1})^*\beta(0), \dots, \widetilde{i}^*(g_{n-1}^{-1})^*\beta(0))$$

is a basis of $T_0^{(1,0)*}\mathbb{C}^{n-1}$. All the g_i's can be chosen in $\mathrm{Comm}(\Gamma) \cap K(\xi, o)$, which is dense in $K(\xi, o)$ by Proposition 3.13.

Then let Γ' be some finite index subgroup of

$$\bigcap_{i=1}^{n-1} g_i \Gamma g_i^{-1} \cap \Gamma$$

which is normal in Γ. We want to show that for every boundary torus T' of $X_{\Gamma'}$ above T and for every point $x' \in T'$ above x, the differential $dA_{\Gamma'}{}_{|T'}(x')$ is injective. Since Γ' is normal in Γ, the automorphism group of $X_{\Gamma'}$ acts transitively on the set of tori above T, thus it is enough to show this property for the torus T' above T which compactifies the cusp associated with the Γ'-orbit of ξ.

The group $\Gamma' \cap [N_\xi, N_\xi]$ is a non-trivial subgroup of $\Gamma \cap [N_\xi, N_\xi]$, it is thus generated by dZ_ξ for some $d \in \mathbb{N}^*$. We then have the following diagram:

$$\begin{array}{ccccccc} \mathbb{H}^n_{\mathbb{C}}/\langle dZ_\xi\rangle & \xrightarrow{\mathcal{L}^{(d)}} & \Omega^{(d)} & \hookrightarrow & \widehat{\Omega^{(d)}} & \xhookleftarrow{\widetilde{i}^{(d)}} & \mathbb{C}^{n-1} \\ \downarrow & & & & \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \mathrm{Id}} \\ \mathbb{H}^n_{\mathbb{C}}/\langle Z_\xi\rangle & \xrightarrow{\mathcal{L}} & \Omega & \hookrightarrow & \widehat{\Omega} & \xhookleftarrow{\widetilde{i}} & \mathbb{C}^{n-1} \\ \downarrow & & & & \downarrow & & \downarrow \\ \mathbb{H}^n_{\mathbb{C}}/\Gamma & & \hookrightarrow & & X_\Gamma & \xhookleftarrow{i} & T, \end{array} \tag{3.7}$$

where

$$\Omega^{(d)} := \left\{ (b, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \mid 0 < |b| < \exp\left(\frac{-\pi \|v\|^2}{dl}\right) \right\},$$

$$\widehat{\Omega^{(d)}} := \Omega^{(d)} \cup (\{0\} \times \mathbb{C}^{n-1}) \qquad \text{and}$$

$$\Psi \colon (b, v) \mapsto (b^d, v).$$

Let $\delta := \Psi^*\beta$, and define $\delta_i := (g_i^{-1})^*\delta$ for $i \in \{1, \ldots, n-1\}$. We claim that $\delta_1, \ldots, \delta_{n-1}$ are the pullback of holomorphic forms $\epsilon_1, \ldots, \epsilon_{n-1}$ on $X_{\Gamma'}$, and that denoting $T' \overset{i'}{\hookrightarrow} X_{\Gamma'}$ the inclusion of T' in $X_{\Gamma'}$, the family $e'' := (i'^*\epsilon_1(x'), \ldots, i'^*\epsilon_{n-1}(x'))$ is a basis of $T_{x'}^{(1,0)*}T'$. This will imply that the differential $dA_{\Gamma'}{}_{|T'}(x')$ is injective.

For all i, the form δ_i is the pullback of a holomorphic form ϵ_i because the element $g_i \in \mathrm{PU}(n,1)$ induces a biholomorphism $\mathbb{H}^n_{\mathbb{C}}/\Gamma \to \mathbb{H}^n_{\mathbb{C}}/g_i\Gamma g_i^{-1}$ which extends to a biholomorphism $\theta_i \colon X_\Gamma \to X_{g_i\Gamma g_i^{-1}}$. Noticing that g_i acts on $\widehat{\Omega^{(d)}}$ and that there is a ramified covering $\pi_i \colon X_{\Gamma'} \to X_{g_i\Gamma g_i^{-1}}$, the commutativity of the following diagram entails that δ_i is the pullback of the form $\epsilon_i := \pi_i^*(\theta_i)_*\alpha$:

$$\begin{array}{ccccc}
\widehat{\Omega^{(d)}} & \xrightarrow{g_i\cdot} & \widehat{\Omega^{(d)}} & \longrightarrow & X_{\Gamma'} \\
\downarrow{\scriptstyle \Psi} & & & & \\
\widehat{\Omega} & & & & \downarrow{\scriptstyle \pi_i} \\
\downarrow & & & & \\
X_\Gamma & & \xrightarrow{\quad\theta_i\quad} & & X_{g_i\Gamma g_i^{-1}}.
\end{array}$$

The family e'' is a basis because, up to the cover $\mathbb{C}^{n-1} \to T'$ it can be identified with the family $e' := (\widetilde{i}^{(d)*}\delta_1(0), \ldots, \widetilde{i}^{(d)*}\delta_{n-1}(0))$ which, up to the identity map $\mathbb{C}^{n-1} \to \mathbb{C}^{n-1}$ of the commutative diagram (3.7), is (identified with) e. The latter was constructed to be a basis of $T_0^{(1,0)*}\mathbb{C}^{n-1}$.

We deduce that $dA_{\Gamma'}{}_{|T'}(x')$ is injective. Since $A_{\Gamma'}{}_{|T'}$ is a holomorphic map between tori, it is a fibration, and in particular, its differential is injective at a point if and only if it is injective at any point. Hence $A_{\Gamma'}{}_{|T'}$ is an immersion.

For each virtually Albanese non-constant boundary torus τ_i, there is a finite index subgroup $\Gamma'_i < \Gamma_0$ constructed as above. Let Γ' be the intersection of the Γ'_i. Let Γ'' be a finite index subgroup of Γ', and T a boundary torus of $X_{\Gamma''}$. If T covers a boundary torus of X_{Γ_0} which is not virtually Albanese non-constant, then by definition $A_{\Gamma''}{}_{|T}$ is constant. Otherwise, there exists an intermediate ramified covering $X_{\Gamma''} \to X_{\Gamma'_i} \to X_{\Gamma_0}$ such that the Albanese map of $X_{\Gamma'_i}$ in restriction to a torus below T is an immersion. *A fortiori* the Albanese map of $X_{\Gamma''}$ in restriction to T is an immersion. □

Remark. We outline there a proof of the classical fact that the boundary tori arising from arithmetic toroidal compactifications in dimension $n \geq 3$ are not simple. This proof was explained to us by Matthew Stover. We will use the description of arithmetic lattices given in the proof of Proposition 3.13. Let Γ_0 be an arithmetic lattice in $\mathrm{PU}(n,1)$, Γ be a finite index subgroup of Γ_0 as in the proof of Proposition 3.13 and T be a boundary torus of X_Γ which corresponds to a parabolic point $\xi \in \partial\mathbb{H}^n_{\mathbb{C}}$. There is a basis $f_\xi = (f_1, f_2, \ldots, f_{n+1})$ of $\mathbb{C}^{n+1}$ as in Section 1.2.1.3 with coefficients in ℓ. In this basis, the matrix group that corresponds to the parabolic subgroup of Γ fixing ξ is of the form $\widehat{P} := gPg^{-1}$ with $g \in \mathrm{GL}_{n+1}(\ell)$ and P a subgroup of $\mathrm{GL}_{n+1}(\mathcal{O}_\ell)$. We deduce that $\widehat{P}_0 := \widehat{P} \cap \mathrm{GL}_{n+1}(\mathcal{O}_\ell)$ has finite index in $\widehat{P}$. The quotient of $\widehat{P}_0$ by its center is a lattice Λ in $\mathbb{C}^{n-1}$ which is included in $\mathcal{O}_l^{n-1}$ and therefore $\mathbb{C}^{n-1}/\Lambda$ is isogenous to $(\mathbb{C}/\mathcal{O}_l)^{n-1}$. We deduce that T itself is isogenous to the product of elliptic curves $(\mathbb{C}/\mathcal{O}_l)^{n-1}$. Hence T is not simple.

3.5 The Shafarevich conjecture for the manifold X_Γ

In this section we prove Theorem K, that we first restate here.

Theorem K. *Let Γ_0 be a non-uniform and torsion-free arithmetic lattice in $\mathrm{PU}(n,1)$. Then there exists a finite index subgroup $\Gamma' < \Gamma_0$ such that for any finite index subgroup $\Gamma < \Gamma'$, the universal cover $\widetilde{X_\Gamma}$ of the toroidal compactification X_Γ of $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ is a Stein manifold.*

We describe the global strategy of the proof in the beginning of Subsection 3.5.2. The results which apply in a more general context and may be of independent interest have been gathered in Subsection 3.5.1. In Subsection 3.5.2, we return to toroidal compactifications and show that the Albanese map of X_Γ satisfy the hypotheses of Proposition 3.15, stated and proven in Subsection 3.5.1.

3.5.1 A general result on holomorphic maps whose target have Stein universal cover

The following proposition is stated for projective manifolds which are the domain of a holomorphic map with values in a compact manifold with Stein universal cover. Lemma 3.16 is used to prove the second point of the proposition. We first give a definition which will be used through all Section 3.5.

Definition 3.14. Let V be a subset of a topological space X and $f\colon V \to \mathbb{R}$ be a continuous function. Then f is said to be an *exhaustion relative to* X if for all real number λ, the closure in X of the sublevel set $f^{-1}((-\infty, \lambda))$ is compact.

Proposition 3.15. *Let X be a projective manifold, $\pi\colon \widetilde{X} \to X$ its universal cover and $A\colon X \to Y$ a holomorphic map from X to a complex manifold Y whose universal cover $\widetilde{Y}$ is Stein.*

1. *Define:*
$$Z := \{x \in X \mid A^{-1}(A(x)) \text{ is not finite}\},$$
let U be an open neighborhood of Z and set $\widetilde{U} := \pi^{-1}(U)$. Then Z is an analytic subvariety of X and there is a smooth plurisubharmonic function $\psi\colon \widetilde{X} \to \mathbb{R}_+$, whose vanishing locus is exactly $\widetilde{Z} := \pi^{-1}(Z)$, that is strictly plurisubharmonic on $\widetilde{X} \setminus \widetilde{Z}$, and whose restriction to $\widetilde{X} \setminus \widetilde{U}$ is an exhaustion.

2. *In addition, if there is an open neighborhood V of Z and a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relative to $\widetilde{X}$, then $\widetilde{X}$ is Stein.*

Lemma 3.16. *Let X be a metric space and U, V two open subsets of X with $\overline{U} \subset V$. Suppose that there exist:*

- *A continuous function $\phi\colon V \to (0, +\infty)$, and*
- *A continuous function $\psi\colon X \to \mathbb{R}_+$ such that $\psi^{-1}(\{0\}) \subset U$ and $\psi|_{X\setminus U}$ is an exhaustion.*

Then for any open subset V' containing U whose closure is included in V, there exists an increasing smooth convex function $\chi\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\chi(0) = 0$ and $\chi \circ \psi > \phi$ on $V' \setminus U$.

Proof of Proposition 3.15-1. This proof is mainly taken from [Nap90, Proof of Theorem 6.1], with some modifications to take into account that we work in arbitrary dimension.

Step 1. The set Z is an analytic subvariety of X.

This is classical, see for instance [Fis76, §3.6]. For a more elementary argument using that X is projective, one can also realise X as a submanifold of some projective space $\mathbb{CP}^d$ and then choose for all $x \in X \setminus Z$ a hyperplane H_x of $\mathbb{CP}^d$ which does not intersect the finite set $A^{-1}(A(x))$. Then Z is analytic as a consequence of the following equality, which is easily verified:

$$Z = \bigcap_{x \in X \setminus Z} A^{-1}(A(H_x)).$$

Step 2. For any sequence $(\widetilde{x_k})_k$ of $\widetilde{X} \setminus \widetilde{U}$ with no accumulation point, there is a holomorphic function $f\colon \widetilde{X} \to \mathbb{C}$ which is not bounded on $(\widetilde{x_k})_k$ and vanishes on $\widetilde{Z} := \pi^{-1}(Z)$.

Let $\rho\colon \widetilde{Y} \to Y$ be the universal cover of Y and $\widetilde{A}\colon \widetilde{X} \to \widetilde{Y}$ the lift of A, so that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{A}} & \widetilde{Y} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \rho} \\ X & \xrightarrow{A} & Y. \end{array}$$

If $(\widetilde{A}(\widetilde{x_k}))_k$ has no accumulation point, then as $\widetilde{Y}$ is a Stein manifold, there is a holomorphic function $u\colon \widetilde{Y} \to \mathbb{C}$ which is unbounded on $(\widetilde{A}(\widetilde{x_k}))_k$ and vanishes on the subvariety $\rho^{-1}(A(Z))$. The function $u \circ \widetilde{A}$ has the required properties.

Otherwise, the sequence $\widetilde{y_k} := \widetilde{A}(\widetilde{x_k})$ has an accumulation point, and up to extracting a subsequence, we can assume, on the one hand, that it converges to an element $\widetilde{y_\infty} \in \widetilde{Y}$, and on the other hand that the sequence $x_k := \pi(\widetilde{x_k})$ converges to a limit $x_\infty \in X \setminus Z$. We claim that there exists an ample line bundle $L \to X$ such that:

- There exists a holomorphic section $t \in H^0(\widetilde{X}, \pi^* L)$ which is unbounded on $(\widetilde{x_k})_k$ with respect to $\pi^* h$ for some Hermitian metric h on L.
- There exists a holomorphic section $s \in H^0(X, L)$ whose vanishing locus $S \subset X$ satisfies:
  - ⋄ $S \cap A^{-1}(A(x_\infty)) = \emptyset$, in other words $A(x_\infty) \notin A(S)$. In particular, $\pi^* s(\widetilde{x_k}) \longrightarrow s(x_\infty) \neq 0$.
  - ⋄ The complement of S does not contain a compact subvariety of positive dimension of X.

Indeed, since X is a projective manifold, it can be realized as a submanifold of some projective space $\mathbb{CP}^d$. Let $H \subset \mathbb{CP}^d$ be a projective hyperplane disjoint from the finite set $A^{-1}(A(x_\infty))$. Then $\mathcal{O}(1)|_X$ is positive so by [Nap90, Corollary 4.3], for a large enough integer p, the bundle $L := \mathcal{O}(p)|_X$ is sufficiently positive so that there exists a holomorphic section $t \in H^0(\widetilde{X}, \pi^* L)$ unbounded on $(\widetilde{x_k})_k$. For the second point, let $\widetilde{s_0}$ be a global holomorphic section of $\mathcal{O}(1)$ whose vanishing locus is H. Then the vanishing locus of the section $\widetilde{s} := \widetilde{s_0}^{\otimes p}$ of $\mathcal{O}(p)$ is H and its complement, biholomorphic to $\mathbb{C}^d$, is a Stein manifold and therefore does not contain any compact subvariety of positive dimension. The restriction $s \in H^0(X, L)$ of $\widetilde{s}$ to X, whose vanishing locus is $S := X \cap H$ has the required properties. Set also $\widetilde{S} := \pi^{-1}(S)$.

The set $E := \rho^{-1}(A(S))$ is an analytic subvariety of $\widetilde{Y}$, which is equal to $\rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$. Since $A(x_\infty) \notin A(S)$, the point $\widetilde{y_\infty}$, whose image by ρ is $A(x_\infty)$, does not belong to E. We deduce that there exists a holomorphic function $u\colon \widetilde{Y} \to \mathbb{C}$ which vanishes on E but does not vanish on $\widetilde{y_\infty}$. Then, E contains $\widetilde{A}(\widetilde{S})$, and, we will show that it also contains $\widetilde{A}(\widetilde{Z})$. Let $\widetilde{x} \in \widetilde{Z}$ and

$x := \pi(\widetilde{x})$. By definition of Z, the analytic subvariety $A^{-1}(A(x))$ is not finite, so has a non-empty intersection with S. In other words, $\rho \circ \widetilde{A}(\widetilde{x}) = A(x) \in A(S)$, so $\widetilde{A}(\widetilde{x}) \in E$. Thus, $u \circ \widetilde{A}$ vanishes on $\widetilde{S} \cup \widetilde{Z}$. As the divisor of $\pi^* s$ is $p\widetilde{S}$, we deduce that the meromorphic function

$$f := (u \circ \widetilde{A})^{p+1} \times (t/\pi^* s)$$

is in fact holomorphic. Since t is not bounded on $(\widetilde{x_k})_k$, and the sequences $u(\widetilde{y_k})$ and $\pi^* s(\widetilde{x_k})$ converge respectively to nonzero limits $u(\widetilde{y_\infty})$ and $s(x_\infty)$, the function f is not bounded on $(\widetilde{x_k})_k$. Finally it is divisible as a holomorphic function by $u \circ \widetilde{A}$, thus it vanishes on $\widetilde{Z}$.

Step 3. For any point $\widetilde{x_0} \in \widetilde{X} \setminus \widetilde{Z}$, there exists a holomorphic function $F \colon \widetilde{X} \to \mathbb{C}^n$ which vanishes on $\widetilde{Z}$, does not vanish at $\widetilde{x_0}$ and whose differential at $\widetilde{x_0}$ is invertible ($n = \dim X$).

Let us write $x_0 := \pi(\widetilde{x_0})$. We assert that there exists an ample line bundle $L \to X$ and holomorphic sections $s, t_1, \ldots, t_n$ of L such that:

- The section s does not vanish on $A^{-1}(A(x_0))$, and in particular does not vanish at x_0 so that the meromorphic functions t_i/s are holomorphic on a neighbourhood of x_0.
- In a neighbourhood of x_0, the holomorphic functions $(\frac{t_1}{s}, \ldots, \frac{t_n}{s})$ do not vanish and define coordinates of X.
- The complement of the vanishing locus $S \subset X$ of s does not contain a compact subvariety of positive dimension of X.

Indeed, embedding X in $\mathbb{CP}^d$ we choose $L := \mathcal{O}(1)|_X$, and $s := \widetilde{s}|_X$ where $\widetilde{s}$ is a section of $\mathcal{O}(1)$ whose vanishing hyperplane H is disjoint from $A^{-1}(A(x_0))$. On the complement of H, the section $\widetilde{s}$ trivializes $\mathcal{O}(1)$, therefore it makes sense to speak of the differential of a section $\widetilde{t}$ of $\mathcal{O}(1)$ at a point of the complement of H: it is the differential of the function $\widetilde{t}/\widetilde{s}$. There are homogeneous coordinates $[z_1 : \ldots : z_{d+1}]$ for which $\widetilde{s}$ identifies with the linear form z_{d+1}, the point x_0 has coordinates $[1 : 1 : \ldots : 1]$ and, in affine coordinates $(z_1, \ldots, z_d) \mapsto [z_1 : \ldots : z_d : 1]$, the tangent space of X at x_0 is spanned by $(\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n})$. The linear forms $z_1, \ldots, z_n$ identify with global sections $\widetilde{t_1}, \ldots, \widetilde{t_n}$ of $\mathcal{O}(1)$. Let $t_1, \ldots, t_n$ denote the restriction of these sections to X. By construction, the holomorphic functions $(\frac{t_1}{s}, \ldots, \frac{t_n}{s})$ do not vanish and define coordinates of X.

Let $S := X \cap H$ be the vanishing locus of s, and $\widetilde{S} := \pi^{-1}(S)$. By construction, the functions $\pi^*(t_i/s)$ are meromorphic on $\widetilde{X}$, and their divisors satisfy $\mathrm{div}(\pi^*(t_i/s)) \geq -\widetilde{S}$. As above, $E := \rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$ is an analytic subvariety of $\widetilde{Y}$ containing $\widetilde{A}(\widetilde{S}) \cup \widetilde{A}(\widetilde{Z})$ and not containing $\widetilde{A}(\widetilde{x_0})$. By Cartan's Theorem B, there exists a holomorphic function $u \colon \widetilde{Y} \to \mathbb{C}$ which vanishes on $E := \rho^{-1}(\rho \circ \widetilde{A}(\widetilde{S}))$, does not vanish at $\widetilde{A}(\widetilde{x_0})$ and whose differential at $\widetilde{A}(\widetilde{x_0})$ is zero. In particular, $u \circ \widetilde{A}$ vanishes on $\widetilde{S} \cup \widetilde{Z}$ but not at $\widetilde{x_0}$. Let us then set

$$f_i := (u \circ \widetilde{A})^2 \times \pi^* \frac{t_i}{s}.$$

Then $F := (f_1, \ldots, f_n)$ is holomorphic. We assert that F vanishes on $\widetilde{Z}$, does not vanish at $\widetilde{x_0}$ and that its differential is invertible at $\widetilde{x_0}$. Indeed, for all i, f_i is divisible by $u \circ \widetilde{A}$, thus it vanishes on $\widetilde{Z}$. However f_i does not vanish at $\widetilde{x_0}$ by construction. Finally, the differential of $u \circ \widetilde{A}$ is zero at $\widetilde{x_0}$ so:

$$(df_i)_{\widetilde{x_0}} = u(\widetilde{A}(\widetilde{x_0}))^2 \pi^* d\left(\frac{t_i}{s}\right)_{x_0}.$$

Therefore, the differential of F is invertible at $\widetilde{x_0}$.

Step 4. Construction of ψ from these two statements: we refer to [Nap90, pp. 470-472]. □

To prove the second point of Proposition 3.15, we need Lemma 3.16, that we prove here.

Proof of Lemma 3.16. First, we claim that $m := \inf_{V' \setminus U}(\psi)$ is positive. Indeed, seeking a contradiction, suppose that $m = 0$. Then there is a sequence (v_n) of elements of $V' \setminus U$ such that $\psi(v_n) \to 0$. By assumption, $\psi|_{X\setminus U}$ is an exhaustion, thus up to taking some subsequence, we can assume that (v_n) converges toward a limit $v_\infty \in X \setminus U$ such that $\psi(v_\infty) = 0$. This contradicts the hypothesis $\psi^{-1}(\{0\}) \subset U$.

If ϕ is bounded on $V' \setminus U$ by some constant C, then $\chi(x) := \frac{C+1}{m}x$ is suitable. Assume now that ϕ is unbounded on $V' \setminus U$, and for all $p \in \mathbb{N}^*$, define the set $A_p := \{x \in V' \setminus U \mid p-1 \leq \phi(x) < p\}$. Enumerate the integers p such that $A_p \neq \emptyset$ as an increasing sequence of integers $(p_k)_{k\in\mathbb{N}}$. For all integer k, let a_k be the infimum of the function $\psi|_{\overline{A_{p_k}}}$, and $x_k \in \overline{V'} \setminus U$ be a point such that $\psi(x_k) \leq a_k + 1$. We claim that $a_k \underset{k\to+\infty}{\longrightarrow} +\infty$. Indeed, seeking a contradiction, suppose that there is a subsequence $(a_{k_j})_{j\in\mathbb{N}^*}$ of (a_k), bounded by some constant M. Then the sublevel $\{\psi|_{X\setminus U} \leq M+1\}$ is compact and contains x_{k_j} for all j so up to a subsequence, we can assume that x_{k_j} converges to a limit $\ell \in \overline{V'}$. We obtain a contradiction by noticing that $\phi(x_{k_j}) \to \phi(\ell)$ and $\phi(x_{k_j}) \geq p_{k_j} - 1 \to +\infty$.

Thus for all integers $n \in \mathbb{N}^*$ there exists $k_n \in \mathbb{N}^*$ such that $\forall k \geq k_n, \frac{a_k}{m} \geq n$. We can choose $k_1 = 1$ and arrange so that the sequence $(k_n)_{n\in\mathbb{N}^*}$ is increasing. Let χ be a smooth increasing convex function such that $\chi(0) = 0$ and $\chi(nm) \geq p_{k_{n+1}}$ $\forall n \in \mathbb{N}^*$. For instance, we can choose for χ a smooth approximation of the piecewise linear function χ_0 with $\chi_0(0) = 0$ and whose leading rate between mn and $m(n+1)$ is the sequence r_n defined recursively by $r_0 = \frac{p_{k_2} - p_{k_1}}{m} + 1$ and $r_{n+1} = \max(\frac{p_{k_{n+1}} - p_{k_n}}{m}, r_n) + 1$.

We claim that $\chi \circ \psi > \phi$ on $V' \setminus U$. Indeed, for all $k \in \mathbb{N}^*$, we can find some $n \in \mathbb{N}^*$ such that $k_n \leq k < k_{n+1}$. Then because $k \geq k_n$, we find $a_k \geq nm$ so $\forall x \in A_{p_k}, \psi(x) \geq a_k \geq nm$. We deduce that

$$\forall x \in A_{p_k}, \chi \circ \psi(x) \geq \chi(nm) \geq p_{k_{n+1}} > \phi(x).$$

□

Proof of Proposition 3.15-2. By assumption, there is an open neighborhood V of Z and a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relative to $\widetilde{X}$. Let U be an open neighbourhood of Z whose closure is included in V, and $\psi\colon \widetilde{X} \to \mathbb{R}$ be the function given by the first point of the Proposition. Write $\widetilde{Z} := \pi^{-1}(Z)$.

After replacing $\widetilde{V}$ by a smaller open subset $\widetilde{V'}$, let χ be an increasing smooth convex function, given by Lemma 3.16, such that $\chi(0) = 0$ and $\chi \circ \psi > \phi + 2$ on $\widetilde{V'} \setminus \widetilde{U}$. Set $\phi_2 := \phi + 1$ and $\psi_2 := \chi \circ \psi$. Setting $\eta = \frac{1}{2}$, let $M_{(\eta,\eta)}$ be the regularized maximum function defined in [Dem, Lemma I.5.18]. Recall that this is a smooth symmetric function on $\mathbb{R}^2$ with the following properties: $\max(x, y) \leq M_{(\eta,\eta)}(x, y)$, and $M_{(\eta,\eta)}(x, y) = y$ whenever $y \geq x + 2\eta$. Also, $M_{(\eta,\eta)}(u_1, u_2)$ is (strictly) plurisubharmonic whenever u_1, u_2 are (strictly) plurisubharmonic functions. Now let $\lambda\colon \widetilde{X} \to \mathbb{R}$ be the function defined by

$$\lambda(x) := \begin{cases} M_{(\eta,\eta)}(\phi_2(x), \psi_2(x)) & \text{if } x \in \widetilde{V'}, \\ \psi_2(x) & \text{if } x \in \widetilde{X} \setminus \widetilde{V'}. \end{cases}$$

Notice that $\psi_2 \geq \phi_2 + 2\eta$ on $\widetilde{V'} \setminus \widetilde{U}$ and $\phi_2 \geq \psi_2 + 2\eta$ on some neighbourhood of $\widetilde{Z}$. Hence λ is a smooth strictly plurisubharmonic function. For any real number C, we have

$$\{x \in \widetilde{X} \mid \lambda(x) \leq C\} \subset \{x \in \widetilde{X} \setminus \widetilde{U} \mid \psi_2(x) \leq C\} \cup \{x \in \widetilde{V} \mid \phi_2(x) \leq C\}.$$

The right-hand side being compact, we infer that λ is an exhaustion, which implies that $\widetilde{X}$ is a Stein manifold. □

3.5.2 A strictly plurisubharmonic exhaustion function on the universal cover of X_Γ

From now on, we fix a non-uniform torsion-free lattice Γ of $\mathrm{PU}(n,1)$ satisfying the following properties: the space $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ admits a toroidal compactification X_Γ with π_1-injective boundary tori, the Albanese map $A_\Gamma \colon X_\Gamma \to \mathrm{Alb}(X_\Gamma)$ is an immersion on the open set $\mathbb{H}^n_{\mathbb{C}}/\Gamma$ and for any boundary torus T of X_Γ, $A_{\Gamma|T}$ is either an immersion or a constant.

Theorem 3.17. *For a lattice Γ as above, the universal cover $\widetilde{X_\Gamma}$ of X_Γ is Stein.*

This statement implies Theorem K: indeed, in Hummel and Schroeder's article, it is noticed that boundary tori are totally geodesic in X_Γ for the metric of nonpositive Riemannian curvature constructed there, and they are thus π_1-injected. See also Theorem H above. In combination with Theorem J, we deduce that for any torsion-free arithmetic lattice Γ_0 of $\mathrm{PU}(n,1)$, there exists a finite index subgroup $\Gamma' < \Gamma_0$ all of whose finite index subgroups satisfy these properties. We will also use that toroidal compactifications are projective manifolds.

Global strategy of proof. Let us denote by $\pi \colon \widetilde{X_\Gamma} \to X_\Gamma$ the universal cover of X_Γ. We can construct holomorphic functions on $\widetilde{X_\Gamma}$ of the form $f := gh$ where h is a meromorphic function obtained by the theorem of holomorphic convexity with respect to a line bundle [Nap90, Corollary 4.3] and g is a holomorphic function which kills the poles of h, obtained by post-composing the lift of the Albanese map of X_Γ by a holomorphic function from $\mathbb{C}^{b_1(X_\Gamma)/2}$ to $\mathbb{C}$. This strategy gives enough holomorphic functions to construct a strictly plurisubharmonic exhaustion ψ on $\widetilde{X_\Gamma} \setminus \pi^{-1}(U)$ where U is an open neighborhood of the set Z consisting of all points $x \in X_\Gamma$ whose fiber $A_\Gamma^{-1}(A_\Gamma(x))$ is not finite (Proposition 3.15-1). We also construct by hand a strictly plurisubharmonic exhaustion function ϕ on the lift of an open neighborhood of Z (Lemmas 3.18, 3.19) and finally we glue ϕ and ψ (Lemma 3.16) to obtain a strictly plurisubharmonic exhaustion function on $\widetilde{X_\Gamma}$ (Proposition 3.15-2). We use the fact that A_Γ restricted to any boundary torus T is either an immersion or a constant in order to get a nice description of Z (Lemma 3.19 and its proof). □

Proof of Theorem 3.17. The theorem is proven once we have shown that the Albanese map of X_Γ satisfies all the hypotheses of Proposition 3.15. This is the content of Lemma 3.19 below, whose proof uses Lemmas 3.18. □

Lemma 3.18 does not use the assumptions on the Albanese map which are described in the beginning of the subsection, contrary to Lemma 3.19. Both statements use the notion of relative exhaustion thas been defined in Subsection 3.5.1.

Lemma 3.18. *Let T be a boundary torus of X_Γ. If V is a sufficiently small open neighbourhood of T, its preimage $\widetilde{V} := \pi^{-1}(V)$ is a disjoint union of neighbourhoods of (copies of) $\mathbb{C}^{n-1}$. We can choose V so that each of these open neighbourhood admits a strictly plurisubharmonic function which is of exhaustion relative to $\widetilde{X_\Gamma}$.*

Lemma 3.19. *Define*

$$Z := \{x \in X_\Gamma \mid A_\Gamma^{-1}(A_\Gamma(x)) \text{ is not finite}\}.$$

Then the set Z is a finite number of fibers of the map A_Γ, and it consists of some boundary tori together with a finite number of points. In particular, there exists a neighbourhood V of Z as well as a smooth strictly plurisubharmonic function ϕ on $\widetilde{V} := \pi^{-1}(V)$ which is an exhaustion relative to $\widetilde{X_\Gamma}$.

Proof of Lemma 3.18. By hypothesis, for any boundary torus T of X_Γ, the map $\pi_1(T) \to \pi_1(X_\Gamma)$ induced by the inclusion $T \hookrightarrow X_\Gamma$ is injective, therefore the preimage of T in the universal cover $\widetilde{X_\Gamma}$ of X_Γ is a disjoint union of copies of $\mathbb{C}^{n-1}$.

Let $p\colon \mathbb{C}^{n-1} \to T$ be the universal cover of a boundary torus T of X_Γ, V a neighbourhood of T identified with the unit disk bundle of a holomorphic line bundle on T endowed with a Hermitian metric h as in Proposition 3.1, and $F\colon V \to T$ the map defining this bundle. Then we have the following commutative diagram:

$$\begin{array}{ccc} \widetilde{V} & \xrightarrow{\ \pi\ } & V \\ \big\downarrow{\scriptstyle \widetilde{F}} & & \big\downarrow{\scriptstyle F} \\ \mathbb{C}^{n-1} & \xrightarrow{\ p\ } & T, \end{array}$$

where $\pi\colon \widetilde{V} \to V$ is the universal cover of V and $\widetilde{F}$ the lift of F. We identify $\widetilde{V}$ with a connected component of the preimage of V in $\widetilde{X_\Gamma}$ and $\mathbb{C}^{n-1}$ with the connected component of the preimage of T in $\widetilde{X_\Gamma}$ contained in $\widetilde{V}$. We will show that some neighborhood of the zero section in $\widetilde{V}$ has a strictly plurisubharmonic function ϕ which is an exhaution relatively to $\widetilde{X_\Gamma}$. This function ϕ will be the sum of the pullback by $\widetilde{F}$ of a strictly plurisubharmonic exhaustion function on $\mathbb{C}^{n-1}$, for instance $z \mapsto \|z\|^2$, and a bounded function on the fibers, defined to be the pullback by π of the squared hermitian norm on V, which will be strictly plurisubharmonic in a neighborhood of the zero section. Explicitely, for $\widetilde{v} \in \widetilde{V}$, let us put $\phi(\widetilde{v}) := \|\widetilde{F}(\widetilde{v})\|^2 + N(\pi(\widetilde{v}))$, where $N\colon V \to \mathbb{R}_+$ is the function $v \mapsto h(v, v)$. Then the function ϕ is an exhaustion relative to $\widetilde{X_\Gamma}$.

We now assert that there is an open neighborhood V' of Z, included in V, such that ϕ is strictly plurisubharmonic on $\pi^{-1}(V')$. First, working in coordinates, it is easily verified that the map

$$(a, v) \in \mathbb{C} \times \mathbb{C}^{n-1} \longmapsto \|v\|^2 + \exp(h(v))|a|^2$$

is strictly plurisubharmonic at all points of the form $(0, v)$. Then since $i\partial\bar{\partial}\phi$ is invariant under the action of $\pi_1(T)$ on $\widetilde{V}$, which is co-compact on some closed neighbourhood of $\mathbb{C}^{n-1}$, we deduce that ϕ is strictly plurisubharmonic on a neighbourhood of $\mathbb{C}^{n-1}$ of the form $\pi^{-1}(V')$ for some open subset $V' \subset V$. □

Proof of Lemma 3.19. Let $T_1, \dots, T_k$ be the boundary tori of X_Γ on which the Albanese map A_Γ is constant, of values $a_1, \dots, a_k$. We claim that $Z = A_\Gamma^{-1}(\{a_1, \dots, a_k\})$, which implies that it is an analytic set. Indeed, if $x \in Z$, there exists a smooth immersed curve $z \mapsto \gamma(z)$ in $A_\Gamma^{-1}(A_\Gamma(x))$. For all z, $dA_\Gamma \cdot \gamma'(z) = 0$ so γ is included in a torus. The Albanese map is an immersion in restriction to all tori except $T_1, \dots, T_k$ so γ must be included in one of the T_i, and $A_\Gamma(x) = a_i$. This proves one inclusion, the other being immediate. In particular, any smooth curve included in Z is either constant or included in one of the tori $T_1, \dots, T_k$ on which the Albanese map A_Γ is constant. We deduce that Z is the union of the tori $T_1, \dots, T_k$ with a discrete subset of $X_\Gamma \setminus \bigcup_{i=1}^k T_i$. This discrete subset has to be finite, since a compact analytic set has finitely many irreducible components. The last part of the lemma then follows from Lemma 3.18 and the fact that each point has a Stein neighborhood. □

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